

MATHEMATICS

JEE (Mains + Advanced)

Function

- Theory
- DPP
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- Answer Key

Function

1. NUMBERSYSTEM

1.1 Natural Numbers

Counting numbers 1, 2, 3, 4, 5,..... are known as natural numbers. The set of all natural numbers can be represent by

$$N = \{1, 2, 3, 4, 5, \dots\}$$

1.2 Whole Numbers

If we include 0 among the natural numbers, then the numbers 0, 1, 2, 3, 4, 5,..... are called whole numbers. The set of whole numbers can be represented by

$$W = \{0, 1, 2, 3, 4, 5, \dots\}$$

Clearly, every natural number is a whole number but 0 is a whole number which is not a natural number.

1.3 Integers

All counting numbers and their negatives including zero are known as integers. The set of integers can be represented by

$$Z = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

1.4 Positive integers

The set $I^+ = \{1, 2, 3, 4, \dots\}$ is the set of all positive integers, Clearly, positive integers and natural numbers are synonyms.

1.5 Negative integers

The set $I^- = \{-1, -2, -3, \dots\}$ is the set of all negative integers. 0 is neither positive nor negative.

1.6 Non-negative integers

The set $\{0, 1, 2, 3, \dots\}$ is the set of all non-negative integers.

1.7 Even or Odd Numbers

Number which are divisible by 2 are called even and which are not divisible by 2 called odd number. In general, even numbers can be represented by $2n$ and odd numbers can be represented by $2n \pm 1$, where n is an integer

Points to Remember

- (i) The sum and product of any number of even numbers is an even number.
- (ii) The difference of two even numbers is an even number.
- (iii) The sum of odd numbers depends on the number of numbers.
 - (a) If the number of numbers is odd, then sum is an odd number.
 - (b) If the number of numbers is even, then sum is an even number.
- (iv) If the product of a certain number is even, then atleast one of the number has to be even.

1.8 Rational Numbers

The numbers of the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$, are known as rational numbers.

e.g. $\frac{4}{7}, \frac{3}{2}, \frac{5}{8}, \frac{0}{1}, -\frac{2}{3}$, etc. The set of all rational numbers is denoted by Q .

$$\text{i.e. } Q = \left\{ x : x = \frac{p}{q}; p, q \in I, q \neq 0 \right\}$$

Since every natural number a can be written as $\frac{a}{1}$, so a is

rational number. Since 0 can be written as $\frac{0}{1}$ and every

non-zero integer ' a ' can be written as $\frac{a}{1}$, so it is also a

rational number.

Every rational number has a peculiar characteristic that when expressed in decimal form is expressible either in terminating decimals or non-terminating repeating decimals.

(a) Terminating Decimal :

Let x be a rational number whose decimal expansion terminates. Then, x can be repressed in the form $\frac{p}{q}$, where

p and q are co-primes, and prime factorizations of q is of the form $2^m \times 5^n$, where m, n are non-negative integers. In this a finite number of digit occurs after decimal.

For example : $\frac{1}{2} = 0.5, \frac{11}{16} = 0.6875, \frac{3}{20} = 0.15$ etc.

(b) Non-Terminating and Repeating (Recurring Decimal)

Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is not of the form $2^m \times 5^n$, where m, n are non-negative integers. Then, x has a decimal expansion which is non-terminating repeating. In this a set of digits or a digit is repeated continuously.

For example : $\frac{2}{3} = 0.6666\dots = 0.\bar{6}$ and $\frac{5}{11} = 0.454545\dots = 0.\overline{45}$

1.9 Irrational numbers

Those numbers which when expressed in decimal form are neither terminating nor repeating decimals are known as irrational numbers,

e.g. $\sqrt{2}, \sqrt{3}, \sqrt{5}, \pi$, etc.

Note That the exact value of π is not $\frac{22}{7} \cdot \frac{22}{7}$ is rational while π is irrational numbers. $\frac{22}{7}$ is approximate value of π . Similarly, 3.14 is not an exact value of it.

1.10 Prime Number

Except 1 each natural number which is divisible by only 1 and itself is called as prime number e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, etc.

1.11 Co-prime

A pair of two natural numbers having no common factor, other than 1, is called a pair of co-primes.

For Ex. (3, 5), (4, 5), (5, 6), (7, 9), (6, 7) etc., are co-primes

1.12 Twin primes

Prime numbers differing by two are called twin primes, e.g. (3, 5), (5, 7), (11, 13) etc are called twin primes.

1.13 Prime triplet

A set of three consecutive primes differing by 2, such as (3, 5, 7) is called a prime triplet.

**“every prime number except 2 is odd
but every odd number need not be prime”**

1.14 Fractions

- (a) Common fraction : Fractions whose denominator is not 10.
- (b) Decimal fraction : Fractions whose denominator is 10 or any power of 10.
- (c) Proper fraction : Numerator < Denominator i.e. $\frac{3}{5}$.
- (d) Improper fraction : Numerator > Denominator i.e. $\frac{5}{3}$.
- (e) Mixed fraction : Consists of integral as well as fractional part i.e. $3\frac{2}{7}$
- (f) Compound fraction : Fraction whose number and denominator themselves are fractions i.e. $\frac{2/3}{5/7}$

1.15 Composite numbers

All natural numbers, which are not prime are composite numbers. If C is the set of composite number than $C = \{4, 6, 8, 9, 10, 12, \dots\}$

1.16 Imaginary numbers

All the numbers whose square is negative are called imaginary numbers. e.g. $3i, -4i, \dots$, where $i = \sqrt{-1}$.

1.17 Complex numbers

The combined form of real and imaginary numbers is known as complex number.

It is denote by $Z = A + iB$, where A is real and B is imaginary part of Z and $A, B \in R$.

2. SET THEORY

2.1 Basic Definition Of Set

A set is a well defined collection of objects. Here term ‘well defined’ means that there is some definite rule on the basis of which one can decide whether an object is in the collection or not. Sets are generally denoted by capital letters e.g., A, B, C, X, Y, Z etc. If an object ‘a’ is in the set A, then we say that ‘a’ is an element of set A or ‘a’ belongs to set A and we write $a \in A$. If ‘a’ is not in set A, then we say that ‘a’ does not belong to set A and we write $a \notin A$.

The collection of cricketers in the world who have played at least five test matches is a set. However, the collection of good cricket players of India is not a set, because the term “good player” is vague and it is not well defined.

Example 1 Which of the following collection is a set ?

- (a) The collection of all girls in you class.
- (b) The collection of intelligent girls in your class.
- (c) The collection of beautiful girls in your class.
- (d) The collection of tall girls in your class.

Ans. (a)

Solution Clearly collections (b), (c) and (d) are not well defined collections. So, they do not form a set

A set is often described in one of the following two ways.

- a. Notation of a set :** Sets are denoted by capital letters like A, B, C or $\{ \}$ and the entries within the bracket are known as elements of set.
- b. Cardinal number of a set :** Cardinal number of a set X is the number of elements of a set X and it is denoted by $n(X)$ e.g. $X = [x_1, x_2, x_3] \therefore n(X) = 3$

2.2 REPRESENTATION OF SETS

a. Set Listing Method (Roster Method) :

In this method a set is described by listing all the elements, separated by commas, within braces $\{ \}$

For example, the set of vowels of English Alphabet may be described as $\{a, e, i, o, u\}$.

The order in which the elements are written in a set makes no difference. Also, the repetition of an element has no effect.

b. Set builder Method (Set Rule Method) :

In this method, a set is described by characterizing property $P(x)$ of its elements x . In such case the set is described by $\{x : P(x) \text{ holds}\}$ or $\{x | P(x) \text{ holds}\}$, which is read as the set of all x such that $P(x)$ holds. The symbol ‘|’ or ‘:’ is read as such that.

The set of all even integers can be written as $E = \{x : x = 2n, n \in \mathbb{Z}\}$

Example 2 Write the solution set of the equation $x^2 + x - 2 = 0$ in roster form.

Solution The given equation can be written as

$$(x - 1)(x + 2) = 0, \text{ i.e., } x = 1, -2$$

Therefore, the solution set of the given equation can be written in roster form as $\{1, -2\}$.

Example 3 Write the set $\{x : x \text{ is a positive integer and } x^2 < 40\}$ in the roster form.

Solution The required numbers are 1, 2, 3, 4, 5, 6. So, the given set in the roster form is $\{1, 2, 3, 4, 5, 6\}$.

Example 4 Write the set $A = \{1, 4, 9, 16, 25, \dots\}$ in set-builder form.

Solution We may write the set A as

$$A = \{x : x \text{ is the square of a natural number}\}$$

Alternatively, we can write

$$A = \{x : x = n^2, \text{ where } n \in \mathbb{N}\}$$

2.3 TYPE OF SETS

I Finite set :

A set X is called a finite set if its element can be listed by counting or labeling with the help of natural numbers and the process terminates at a certain natural number n . i.e. $n(X) = \text{finite number}$ e.g.

- (1) A set of English Alphabets
- (2) Set of soldiers in Indian Army

II Infinite set :

A set whose elements cannot be listed counted by the natural numbers $(1, 2, 3, \dots, n)$ for any number n , is called a infinite set. e.g.

- (1) A set of all points in a plane
- (2) $X = \{x : x \in \mathbb{R}, 0 < x < 0.0001\}$
- (3) $X = \{x : x \in \mathbb{Q}, 0 \leq x \leq 0.0001\}$

III Singleton set :

A set consisting of a single element is called a singleton set. i.e. $n(X) = 1$, e.g.

- (1) $\{x : x \in \mathbb{N}, 1 < x < 3\}$,
- (2) $\{\{\}\}$: Set of null set,
- (3) $\{f\}$ is a set containing alphabet f .

IV Null set :

A set is said to be empty, void or null set if it has no element in it, and it is denoted by ϕ , i.e., X is a null set if $n(X) = 0$,

e.g., (1) $\{x : x \in \mathbb{R} \text{ and } x^2 + 2 = 0\}$

(2) $\{x : x > 1 \text{ but } x < 1/2\}$

(3) $\{x : x \in \mathbb{R}, x^2 < 0\}$

V Equivalent Set :

Two finite sets A and B are equivalent if their cardinal numbers are same i.e. $n(A) = n(B)$.

VI Equal Set :

Two sets A and B are said to be equal if every element of A is a member of B and every element of B is a member of A . i.e. $A = B$, if A and B are equal and $A \neq B$, if they are not equal.

Example 5 State which of the following sets are finite or infinite.

(i) $\{x : x \in \mathbb{N} \text{ and } (x - 1)(x - 2) = 0\}$

(ii) $\{x : x \in \mathbb{N} \text{ and } x^2 = 4\}$

(iii) $\{x : x \in \mathbb{N} \text{ and } 2x - 1 = 0\}$

(iv) $\{x : x \in \mathbb{N} \text{ and } x \text{ is prime}\}$

(v) $\{x : x \in \mathbb{N} \text{ and } x \text{ is odd}\}$

Solution

- (i) Given set = $\{1, 2\}$. Hence, it is finite.
- (ii) Given set = $\{2\}$. Hence, it is finite.
- (iii) Given set = ϕ . Hence, it is finite.
- (iv) The given set is the set of all prime numbers and since set of prime numbers is infinite. Hence, the given set is infinite.
- (v) Since there are infinite number of odd numbers, hence, the given set is infinite.

VII UNIVERSAL SET

It is a set which includes all the sets under considerations i.e. it is a super set of each of the given set. Thus, a set that contains all sets in a given context is called the universal set. It is denoted by U . e.g.,
If $A = \{1, 2, 3\}$, $B = \{2, 4, 5, 6\}$ and $C = \{1, 3, 5, 7\}$, then $U = \{1, 2, 3, 4, 5, 6, 7\}$ can be taken as the universal set.

VIII SUBSET

A set A is said to be a subset of B if all the elements of A are present in B and is denoted by $A \subset B$ (read as A is subset of B) and symbolically written as : $x \in A \Rightarrow x \in B \Leftrightarrow A \subset B$

Example 6 Let A, B and C be three sets. If $A \in B$ and $B \subset C$, is it true that $A \subset C$? If not, give an example

Solution No. Let $A = \{1\}$, $B = \{\{1\}, 2\}$ and $C = \{\{1\}, 2, 3\}$. Here $A \in B$ as $A = \{1\}$ and $B \subset C$.

But $A \not\subset C$ as $1 \in A$ and $1 \notin C$.

Note that an element of a set can never be a subset of itself.

Number of subsets :

Condiser a set X containing n elements as $\{x_1, x_2, \dots, x_n\}$ then the total number of subsets of $X = 2^n$

Proof: Number of subsets of above set is equal to the number of selections of elements taking any number of them at a time out of the total n elements and it is equal to 2^n
 ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$

Types of Subsets :

A set A is said to be a **proper subset** of a set B if every element of A is an element of B and B has at least one element which is not an element of A and is denoted by $A \subset B$.

The set A itself and the empty set is known as **improper subset** and is denoted as $A \subseteq B$.

e.g. If $X = \{x_1, x_2, \dots, x_n\}$ then total number of proper sets = $2^n - 2$ (excluding itself and the null set). The statement $A \subset B$ can be written as $B \supset A$, then B is called the **super set** of A and is written as $B \supset A$.

IX POWER SET

The collection of all subsets of set A is called the power set of A and is denoted by $P(A)$
 i.e. $P(A) = \{x : x \text{ is a subset of } A\}$.

If $X = \{x_1, x_2, x_3, \dots, x_n\}$ then $n(P(X)) = 2^n$; $n(P(P(X))) = 2^{2^n}$.

Example 7 Find the pairs of equal sets, if any, give reasons:

- $A = \{0\}, B = \{x : x > 15 \text{ and } x < 5\}$,
- $C = \{x : x - 5 = 0\}, D = \{x : x^2 = 25\}$,
- $E = \{x : x \text{ is an integral positive root of the equation } x^2 - 2x - 15 = 0\}$.

Solution Since $0 \in A$ and 0 does not belong to any of the sets B, C, D and E , it follows that, $A \neq B, A \neq C, A \neq D, A \neq E$. Since $B = \emptyset$ but none of the other sets are empty. Therefore $B \neq C, B \neq D$ and $B \neq E$. Also $C = \{5\}$ but $-5 \in D$, hence $C \neq D$. Since $E = \{5\}, C = E$. Further, $D = \{-5, 5\}$ and $E = \{5\}$, we find that, $D \neq E$. Thus, the only pair of equal sets is C and E .

Example 8 Consider the sets $\phi, A = \{1, 3\}, B = \{1, 5, 9\}, C = \{1, 3, 5, 7, 9\}$. Insert the symbol, \subset or $\not\subset$ between each of the following pair of sets :

- (i) $\phi \dots B$ (ii) $A \dots B$ (iii) $A \dots C$ (iv) $B \dots C$

Solution (i) $\phi \subset B$ as ϕ is a subset of every set.
 (ii) $A \not\subset B$ as $3 \in A$ and $3 \notin B$
 (iii) $A \subset C$ as $1, 3 \in A$ also belongs to C
 (iv) $B \subset C$ as each element of B is also an element of C .

2.4 INTERVALS

The set of numbers any two real numbers is called interval. The following are the types of interval.

I Closed Interval

$x \in [a, b] \equiv \{x : a \leq x \leq b\}$



II Open Interval

$x \in (a, b) \text{ or }]a, b[\equiv \{x : a < x < b\}$



III Semi Open or Semi-Closed Interval

$x \in [a, b[\text{ or }]a, b) = \{x : a \leq x < b\}$



$x \in]a, b] \text{ or } (a, b] = \{x : a < x \leq b\}$



Example 9 Write the following as intervals :

- (i) $\{x : x \in \mathbb{R}, -4 < x \leq 6\}$
- (ii) $\{x : x \in \mathbb{R}, -12 < x < -10\}$
- (iii) $\{x : x \in \mathbb{R}, 0 \leq x < 7\}$
- (iv) $\{x : x \in \mathbb{R}, 3 \leq x \leq 4\}$

Solution (i) $(-4, 6]$ (ii) $(-12, -10)$
 (iii) $[0, 7)$ (iv) $[3, 4]$

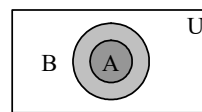
Example 10 Write the following intervals in set-builder form :

- (i) $(-3, 0)$ (ii) $[6, 12]$
- (iii) $(6, 12]$ (iv) $[-23, 5)$

Solution (i) $\{x : x \in \mathbb{R}, -3 < x < 0\}$
 (ii) $\{x : x \in \mathbb{R}, 6 \leq x \leq 12\}$
 (iii) $\{x : x \in \mathbb{R}, 6 < x \leq 12\}$
 (iv) $\{x : x \in \mathbb{R}, -23 \leq x < 5\}$

2.5 VENN (EULER) DIAGRAMS

The diagrams drawn to represent sets are called Venn diagram or Euler-Venn diagrams. Here we represent the universal U as set of all points within rectangle and the subset A of the set U is represented by the interior of a circle. If a set A is a subset of a set B , then the circle representing A is drawn inside the circle representing B . If A and B are not equal but they have some common elements, then to represent A and B by two intersecting circles. E.g. If A is subset of B then it is represented diagrammatically in fig.



2.6 OPERATIONS ON SETS

I Union of sets :

If A and B are two sets then union (\cup) of A and B is the set of all those elements which belong either to A or to B or to both A and B . It is also defined as $A \cup B = \{x : x \in A \text{ or } x \in B\}$. It is represented through Venn diagram in fig.1 & fig.2

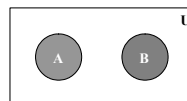


Fig. (1)

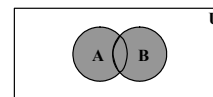


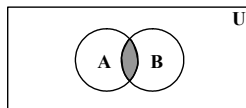
Fig. (2)

Example 11 If $A = \{x : x = 2n + 1, n \in \mathbb{Z}\}$ and $B = \{x : x = 2n, n \in \mathbb{Z}\}$, then find the value of $A \cup B$.

Solution We have,
 $A \cup B = \{x : x = 2n + 1 \text{ or } x = 2n, n \in \mathbb{Z}\}$
 $= \{x : x \text{ is an integer}\} = \mathbb{Z}$

II Intersection of sets :

If A and B are two sets then intersection (\cap) of A and B is the set of all those elements which belong to both A and B. It is also defined as $A \cap B = \{x : x \in A \text{ and } x \in B\}$ represented in Venn diagram (see fig.)



Example 12 If $A = \{x : x = 4n, n \in \mathbb{Z}\}$ and $B = \{x : x = 6n, n \in \mathbb{Z}\}$, then find the value of $A \cap B$.

Solution We have,
 $x \in A \cap B$
 $\Rightarrow x = 4n \text{ and } x = 6n, n \in \mathbb{Z}$
 $\Rightarrow x \text{ is a multiple of 4 and } x \text{ is a multiple of 6}$
 $\Rightarrow x \text{ is a multiple of 4 and 6 both}$
 $\Rightarrow x \text{ is a multiple of 12}$
 $\Rightarrow x = 12n, n \in \mathbb{Z}$
Hence, $A \cap B = \{x : x = 12n, n \in \mathbb{Z}\}$

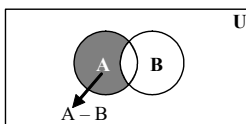
NOTE

Sets A and B are said to be disjoint iff A and B have no common element or $A \cap B = \phi$. If $A \cap B \neq \phi$ then A and B are said to be intersecting or overlapping sets. e.g.

- (i) If $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$ and $C = \{4, 7, 9\}$ then A and B are disjoint set where B and C are intersecting sets.
- (ii) Set of even natural numbers and odd natural numbers are disjoint sets.

III Difference of two sets :

If A and B are two sets then the difference of A and B, is the set of all those elements of A which do not belong to B.

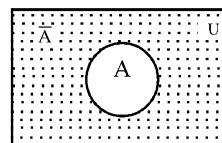


Thus, $A - B = \{x : x \in A \text{ and } x \notin B\}$
or $A - B = \{x \in A ; x \notin B\}$
Clearly $x \in A - B \Leftrightarrow x \in A \text{ and } x \notin B$
It is represented through the Venn diagrams.
The difference $B - A$ is the set of all those elements of B that do not belong to A
i.e. $B - A = \{x : x \in B \text{ and } x \notin A\}$.

IV COMPLEMENTARY SET

Complementary set of a set A is a set containing all those elements of universal set which are not in A. It is denoted

by \bar{A} , A^c or A' . So $A^c = \{x : x \in U \text{ but } x \notin A\}$. e.g. If set $A = \{1, 2, 3, 4, 5\}$ and universal set $U = \{1, 2, 3, 4, \dots, 50\}$ then $\bar{A} = \{6, 7, \dots, 50\}$



- (i) $U' = \phi$
- (ii) $\phi' = U$
- (iii) $(A')' = A$
- (iv) $A \cup A' = U$
- (v) $A \cap A' = \phi$

NOTE

All disjoint sets are not complementary sets but all complementary sets are disjoint.

V Idempotent operation :

For any set A, we have
(i) $A \cup A = A$ and (ii) $A \cap A = A$

Proof :

- (i) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$
- (ii) $A \cap A = \{x : x \in A \text{ \& } x \in A\} = \{x : x \in A\} = A$

VI Identity operation : For any set A, we have

- (i) $A \cup \phi = A$ and
- (ii) $A \cap U = A$ i.e. ϕ and U are identity elements for union and intersection respectively

Proof :

- (i) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\} = A$
- (ii) $A \cap U = \{x : x \in A \text{ and } x \in U\} = \{x : x \in A\} = A$

VII Commutative operation :

For any set A and B, we have

- (i) $A \cup B = B \cup A$ and
- (ii) $A \cap B = B \cap A$
i.e. union and intersection are commutative.

VIII Associative operation :

If A, B and C are any three sets then

- (i) $(A \cup B) \cup C = A \cup (B \cup C)$
- (ii) $(A \cap B) \cap C = A \cap (B \cap C)$
i.e. union and intersection are associative.

IX Distributive operation :

If A, B and C are any three sets then

- (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
i.e. union and intersection are distributive over intersection and union respectively.

X De-Morgan's Principle :

If A and B are any two sets, then

- (i) $(A \cup B)' = A' \cap B'$
- (ii) $(A \cap B)' = A' \cup B'$

Proof:

- (i) Let x be an arbitrary element of $(A \cup B)'$. Then $x \in (A \cup B)'$
 $\Rightarrow x \notin (A \cup B) \Rightarrow x \notin A$ and $x \notin B \Rightarrow x \in A' \cap B'$
 Again let y be an arbitrary element of $A' \cap B'$. Then
 $y \in A' \cap B'$
 $\Rightarrow y \in A'$ and $y \in B' \Rightarrow y \notin A$ and $y \notin B$
 $\Rightarrow y \notin (A \cup B) \Rightarrow y \in (A \cup B)'$
 $\therefore A' \cap B' \subseteq (A \cup B)'$.
 Hence $(A \cup B)' = A' \cap B'$
 Similarly (ii) can be proved.

SOME IMPORTANT RESULTS

If A, B and C are finite sets, and U be the finite universal set, then

- (i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
 (ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets.
 (iii) $n(A - B) = n(A) - n(A \cap B)$
 i.e, $n(A - B) + n(A \cap B) = n(A)$
 (iv) $n(A \cup B \cup C)$
 $= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C)$
 $+ n(A \cap B \cap C)$
 (v) No. of elements in exactly two of the sets A, B, C is
 $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$.

DPP 1

Total Marks : 39

Time 25 min

Instructions

1. Question – 3, 4, 5, 8, 9 marking scheme : +3 for correct answer, –1 in all other cases. [5 × 3 = 15]
 2. Question – 1, 2, 6, 7, 10, 11 marking scheme : +4 for correct answer, 0 in all other cases. [4 × 6 = 24]

1. Write the set $\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7} \right\}$ in the set-builder form.
2. Match each of the set on the left described in the roster form with the same set on the right described in the set-builder form :
- (i) {P, R, I, N, C, A, L} (a) { $x : x$ is a positive integer and is a divisor of 18 }
 (ii) { 0 } (b) { $x : x$ is an integer and $x^2 - 9 = 0$ }
 (iii) { 1, 2, 3, 6, 9, 18 } (c) { $x : x$ is an integer and $x + 1 = 1$ }
 (iv) { 3, –3 } (d) { $x : x$ is a letter of the word PRINCIPAL }
3. If B is the set whose elements are obtained by adding to 1 to each of the even numbers, then the set builder notation of B is
 (A) $B = \{x : x \text{ is even}\}$
 (B) $B = \{x : x \text{ is odd and } x > 1\}$
 (C) $B = \{x : x \text{ is odd and } x \in \mathbb{Z}\}$
 (D) $B = \{x : x \text{ is an integer}\}$

4. Which of the following is the empty set ?

- (A) { $x : x$ is a real number and $x^2 - 1 = 0$ }
 (B) { $x : x$ is a real number and $x^2 + 1 = 0$ }
 (C) { $x : x$ is a real number and $x^2 - 9 = 0$ }
 (D) { $x : x$ is a real number and $x^2 = x + 2$ }

5. Which of the following sets is not finite ?

- (A) { $(x, y) : x^2 + y^2 \leq 1 \leq x + y, x, y \in \mathbb{R}$ }
 (B) { $(x, y) : x^2 + y^2 \leq 1 \leq x + y, x, y \in \mathbb{Z}$ }
 (C) { $(x, y) : x^2 \leq y \leq |x|, x, y \in \mathbb{Z}$ }
 (D) { $(x, y) : x^2 + y^2 = 1, x, y \in \mathbb{Z}$ }

6. Which of the following pairs of sets are equal ? Justify your answer.

- (i) X , the set of letters in “ALLOY” and B , the set of letters in “LOYAL”.
 (ii) $A = \{n : n \in \mathbb{Z} \text{ and } n^2 \leq 4\}$ and $B = \{x : x \in \mathbb{R} \text{ and } x^2 - 3x + 2 = 0\}$.

7. Let $A = \{a, e, i, o, u\}$ and $B = \{a, b, c, d\}$. Is A a subset of B ? No. (Why ?). Is B a subset of A ? No. (Why ?)

8. The collection of intelligent students in a class is :

- (A) a null set
 (B) a singleton set
 (C) a finite set
 (D) not a well defined collection.

9. If $X = \{8^n - 7n - 1 : n \in \mathbb{N}\}$ and $Y = \{49(n - 1) : n \in \mathbb{N}\}$, then

- (A) $X \subset Y$ (B) $Y \subset X$
 (C) $X = Y$ (D) None of these

10. Let $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 3\}$ and $B = \{3, 4, 5\}$. Find A' , B' , $A' \cap B'$, $A \cup B$ and hence show that $(A \cup B)' = A' \cap B'$.

11. Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8\}$. Find $A - B$ and $B - A$.

Result Analysis

1. 30 to 39 Marks : **Advance Level.**
 2. 20 to 29 Marks : **Main Level.**
 3. < 20 Marks : **Please go through this artical again.**

3. INEQUALITIES

3.1 INEQUATIONS

A statement involving variable (s) and the sign of inequality viz., > or, < or, ≥ or, ≤ is called an inequation.

An inequation may contain one or more variables. Also, it may be linear or quadratic or cubic etc.,

If a is a non-zero real number and x, y are variables, then $ax + b < 0$, $ax + b \leq 0$, $ay + b > 0$ and $ay + b \geq 0$ are linear inequations in one variable.

If $a \neq 0, b \neq 0, c$ real numbers and x, y, z are variables, then $ax + by < c$, $ay + bz \leq c$, $ax + by > c$ and $ax + by \geq c$ are linear inequations in two variables.

Inequations $2x - 3 < 0$,

$3y \geq 5$, $\frac{x}{2} - 7 \leq 0$, $\frac{2x-3}{4} \geq -\frac{x}{2} + 7$ and $-2 + t \leq \frac{5}{3}$ are

linear inequations in one variables. Inequations of the form

$\frac{2x^2 - 3}{x - 1} < 4$, $x^2 - 5x + 6 \geq 0$ etc are not linear inequations.

A solution of an inequation is the value(s) of the variable(s) that marks it a true statement.

For example, $x = 9$ is a solution of the inequation

$\frac{3 - 2x}{5} < \frac{x}{3} - 4$, because for $x = 9$ it reduces to $-3 < -1$ which

is a true statement. But, $x = 6$ is not its solution. For $x = 6$,

the inequation reduces to $-\frac{9}{5} < -2$ which is not true. For

any real number x , we have $x^2 + 1 > 0$. So, the solution set of the inequation $x^2 + 1 > 0$ is the set R of all real numbers and the solution set of $x^2 + 1 < 0$ is the null set ϕ .

For solving an inequation, we use the following rules :

RULE 1 Same number may be added to (or subtracted from) both sides of an in equation without changing the sign of inequality.

RULE 2 Both sides of an inequation can be multiplied (or divided) by the same positive real number without changing the sign of inequality. However, the sign of inequality is reversed when both sides of an inequation are multiplied or divided by a negative number.

RULE 3 Any term of an inequation may be taken to the other side with its sign changed without affecting the sign of inequality.

Example 13 Find the solution of the inequation

$$\frac{1}{2} \left(\frac{3}{5}x + 4 \right) \geq \frac{1}{3}(x - 6)$$

Solution We have,

$$\frac{1}{2} \left(\frac{3}{5}x + 4 \right) \geq \frac{1}{3}(x - 6)$$

$$\Rightarrow \frac{1}{2} \left(\frac{3x + 20}{5} \right) \geq \frac{1}{3}(x - 6)$$

$$\Rightarrow \frac{3x + 20}{10} \geq \frac{x - 6}{3}$$

$$\Rightarrow 3(3x + 20) \geq 10(x - 6)$$

[Multiplying both sides by 30, i.e., LCM of 10 and 3]

$$\Rightarrow 9x + 60 \geq 10x - 60$$

$$\Rightarrow 9x - 10x \geq -60 - 60$$

$$\Rightarrow -x \geq -120$$

$$\Rightarrow x \leq 120$$

$$\Rightarrow x \in (-\infty, 120]$$

Hence, the solution set of the given inequation is $(-\infty, 120]$

Remark The solution set of simultaneous inequations is the intersection of their solution sets.

Example 14 If $\frac{5x}{4} + \frac{3x}{8} > \frac{39}{8}$ and

$$\frac{2x - 1}{12} - \frac{x - 1}{3} < \frac{3x + 1}{4}$$
 then find the interval for x .

Solution We have,

$$\frac{5x}{4} + \frac{3x}{8} > \frac{39}{8} \text{ and } \frac{2x - 1}{12} - \frac{x - 1}{3} < \frac{3x + 1}{4}$$

$$\Rightarrow \frac{10x + 3x}{8} > \frac{39}{8} \text{ and } \frac{2x - 1 - 4x + 4}{12} < \frac{3x + 1}{4}$$

$$\Rightarrow \frac{13x}{8} > \frac{39}{8} \text{ and } \frac{-2x + 3}{12} < \frac{3x + 1}{4}$$

$$\Rightarrow 13x > 39 \text{ and } -2x + 3 < 9x + 3$$

$$\Rightarrow x > 3 \text{ and } -11x < 0$$

$$\Rightarrow x > 3 \text{ and } x > 0$$

$$\Rightarrow x \in (3, \infty) \text{ and } x \in (0, \infty) \Rightarrow x \in (3, \infty)$$

3.2 SOLVING RATIONAL ALGEBRAIC INEQUATIONS

If $P(x)$ and $Q(x)$ are polynomials in x , then the inequations

$$\frac{P(x)}{Q(x)} > 0, \frac{P(x)}{Q(x)} < 0, \frac{P(x)}{Q(x)} \geq 0 \text{ and } \frac{P(x)}{Q(x)} \leq 0$$
 are known

as rational algebraic inequations. To solve these inequations we use the sign method as explained in the following algorithm.

ALGORITHM

STEP 1 Obtain $P(x)$ and $Q(x)$

STEP 2 Factorise $P(x)$ and $Q(x)$ into linear factors.

STEP 3 Make the coefficient of x is positive in all factors.

STEP 4 Obtain critical points by equating all factors to zero.

STEP 5 Plot the critical points on the number line. If there are n critical points, they divide the number line into $(n + 1)$ regions.

STEP 6 In the right most region the expression $\frac{P(x)}{Q(x)}$ bears

positive sign and in other regions the expression bears alternate negative and positive signs.

Example 15 Solve the inequality $\frac{x-1}{x} \geq 2$.

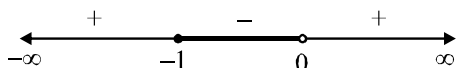
Solution We have,

$$\frac{x-1}{x} \geq 2 \Rightarrow \frac{x-1}{x} - 2 \geq 0 \Rightarrow \frac{-1-x}{x} \geq 0$$

$$\Rightarrow \frac{x+1}{x-0} \leq 0 \quad \dots(i)$$

On equating $x+1$ and $x-0$ to zero, we obtain $x=-1$ and $x=0$ as critical points. These points when plotted on number line divided it into three regions. Marking alternatively positive and negative from the right most region, we obtain

the sign of $\frac{x+1}{x}$ for different values of x as shown in the following figure.



Since the expression in (i) is non-positive. So, the solution set of (i) is $[-1, 0)$.

3.3 Generalized Method of Intervals

Let $F(x) = (x-a_1)^{k_1}(x-a_2)^{k_2} \dots (x-a_{n-1})^{k_{n-1}}(x-a_n)^{k_n}$.

Here, $k_1, k_2, k_n \in Z$ and a_1, a_2, \dots, a_n are fixed real numbers satisfying the condition

$$a_1 < a_2 < a_3 < \dots < a_{n-1} < a_n$$

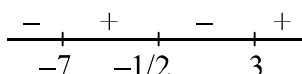
For solving $F(x) > 0$ or $F(x) < 0$, consider the following algorithm.

- We mark the numbers a_1, a_2, \dots, a_n on the number axis and put plus sign in the interval on the right of the largest of these numbers, i.e., on the right of a_n .
- Then we put plus sign in the interval on the left of a_n if k_n is an even number and minus sign if k_n is an odd number. In the next interval, we put a sign according to the following rule:
 - When passing through the point a_{n-1} the polynomial $F(x)$ changes sign if k_{n-1} is an odd number. Then we consider the next interval and put a sign in it using the same rule.
- Thus we consider all the intervals. The solution of the inequality $F(x) > 0$ is the union of all intervals in which we put plus sign and the solution of the inequality $F(x) < 0$ is the union of all intervals in which we minus sign.

Example 16 Solve $(2x+1)(x-3)(x+7) < 0$.

Solution $(2x+1)(x-3)(x+7) < 0$

Sign scheme of $(2x+1)(x-3)(x+7)$ is as follows :

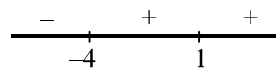


Hence, solution is $(-\infty, -7) \cup (-1/2, 3)$.

Example 17 Solve $(1-x)^2(x+4) < 0$

Solution $(x-1)^2(x+4) < 0 \quad \dots (i)$

Sign scheme of $(x-1)^2(x+4)$ is as follows :



Sign of expression does not change at $x=1$ as $(x-1)$ factor has even power.

Hence, solution of (i) is $x \in (-\infty, -4)$

Example 18 Solve $\frac{2}{x} < 3$

Solution $\frac{2}{x} < 3$

$$\Rightarrow \frac{2}{x} - 3 < 0$$

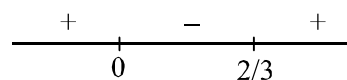
$$\Rightarrow \frac{2-3x}{x} < 0$$

(we cannot cross multiply with x , as x can be negative or positive)

$$\Rightarrow \frac{3x-2}{x} > 0$$

$$\Rightarrow \frac{(x-2/3)}{x} > 0$$

Sign scheme of $\frac{(x-2/3)}{x}$ is as follows :



$$\Rightarrow x \in (-\infty, 0) \cup (2/3, \infty)$$

Example 19 Solve $\frac{2x-3}{3x-5} \geq 3$

Solution $\frac{2x-3}{3x-5} \geq 3$

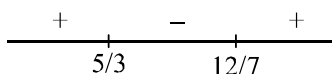
$$\Rightarrow \frac{2x-3}{3x-5} - 3 \geq 0$$

$$\Rightarrow \frac{2x-3-9x+15}{3x-5} \geq 0$$

$$\Rightarrow \frac{-7x+12}{3x-5} \geq 0$$

$$\Rightarrow \frac{7x-12}{3x-5} \leq 0$$

Sign scheme of $\frac{7x-12}{3x-5}$ is as follows :



$$\Rightarrow x \in (5/3, 12/7]$$

$x = 5/3$ is not included in solution as at $x = 5/3$, denominator becomes zero.

$$x \in \left(\frac{5}{3}, \frac{12}{7} \right]$$

DPP 2

Total Marks : 38

Time 25 min

Instructions

1. Question – 4, 5 marking scheme : +3 for correct answer, –1 in all other cases. [3 × 2 = 6]
2. Question – 1, 2, 3, 6, 7, 8, 9, 10 marking scheme : +4 for correct answer, 0 in all other cases. [4 × 8 = 32]

1. Solve $4x + 3 < 6x + 7$.
2. Solve $\frac{5-2x}{3} \leq \frac{x}{6} - 5$.
3. Solve $7x + 3 < 5x + 9$. Show the graph of the solutions on number line.
4. The solution set of the inequation $\frac{x-1}{x-2} > 2$, is
 (A) (2, 3) (B) [2, 3]
 (C) $(-\infty, 2) \cup (3, \infty)$ (D) None of these
5. The set of values of x satisfying the system of inequations $5x + 2 < 3x + 8$ and $\frac{x+2}{x-1} < 4$ is :
 (A) $(-\infty, 1)$ (B) (2, 3)
 (C) $(-\infty, 3)$ (D) $(-\infty, 1) \cup (2, 3)$
6. Solve $-8x \leq 5x - 3 < 7$.
7. Solve $-5 \leq \frac{5-3x}{2} \leq 8$.
8. Solve $\frac{x-1}{x} - \frac{x+1}{x-1} < 2$.
9. Solve $\frac{(x-1)(x-2)(x-3)}{(x+1)(x+2)(x+3)} > 1$.
10. Solve $\frac{(x-4)^{2005} \cdot (x+8)^{2008} \cdot (x+1)}{x^{2006} \cdot (x-2)^3 \cdot (x+3)^5 \cdot (x-6)(x+9)^{2010}} \leq 0$

Result Analysis

1. 30 to 38 Marks : Advance Level.

2. 20 to 29 Marks : Main Level.
3. < 20 Marks : Please go through this artical again.

4. CARTESIAN PRODUCT OF SETS

If A and B are any two non-empty sets, then the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the set A with set B is denoted by $A \times B$. Thus, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

Example 20 If $A = \{1, 2\}$ and $B = \{0, 1\}$, then find $A \times B$.

Solution We have

$$A \times B = \{1, 2\} \times \{0, 1\} = \{(1, 0), (1, 1), (2, 0), (2, 1)\}$$

Example 21 If $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, then find $(A \cap B) \times A$.

Solution We have,

$$A \cap B = \{3\} \text{ and } A = \{1, 2, 3\}$$

$$\therefore (A \cap B) \times A = \{(3, 1), (3, 2), (3, 3)\}$$

Example 22 If $A = \{x \in \mathbb{R} : 0 < x < 1\}$ and

$B = \{x \in \mathbb{R} : -1 < x < 1\}$, then find $A \times B$.

Solution Clearly, A represents the line segment between $(0, 0)$ and $(1, 0)$ and B represents the line segment between $(0, -1)$ and $(0, 1)$.

So, $A \times B$ represents the region in xy -plane lying inside the rectangle having vertices at $(1, 1), (0, 1), (0, -1)$ and $(1, -1)$.

5. RELATIONS

RELATION Let A and B be two sets. Then, a relation R from A to B is a subset of $A \times B$.

If $(a, b) \in R$, then we write aRb which is read as a is related to b by the relation R . If $(a, b) \notin R$, then we write $a \not R b$ and we say that a is not related to b by the relation R .

Example 23 If $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$, then which of the following is a relation from A to B ?

- (A) $R_1 = \{(a, 1), (2, b), (c, 3)\}$
- (B) $R_2 = \{(a, 1), (d, 3), (b, 2), (b, 3)\}$
- (C) $R_3 = \{(1, a), (2, b), (3, c)\}$
- (D) $R_4 = \{(a, 1), (b, 2), (c, 3), (3, d)\}$

Ans. (B)

Solution We observe that $(a, 1), (c, 3) \in A \times B$ but $(2, b) \notin A \times B$.

So, R_1 is not a relation from A to B .

Clearly, $R_2 \subseteq A \times B$. So, it is a relation from A to B .

Since $(2, b) \in R_3$ but $(2, b) \notin A \times B$. So, $R_3 \not\subseteq A \times B$. Hence, it is not a relation from A to B .

We find that $(3, d) \in R_4$ but $(3, d) \notin A \times B$. So, R_4 is not a relation from A to B .

NOTE

If A and B are two finite sets consisting of m and n elements respectively, then $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn} . Hence, 2^{mn} relations can be defined from A to B .

5.1 DOMAIN AND RANGE OF A RELATION

If R is a relation from a set A to a set B , then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R , while the set of all second components or coordinates of the ordered pairs in R is called the range of R . The whole set B is called the codomain of the relation R . Note that $\text{range } R \subseteq \text{codomain}$.

5.2 TYPES OF RELATIONS

I VOID RELATION

Let A be a non-empty set. Then $\phi \subseteq A \times A$ and so it is a relation on set A . This relation is called the void or empty relation on set A .

II UNIVERSAL RELATION

Let A be a non-empty set. Then, $A \times A$ is known as the universal relation on set A .

III IDENTITY RELATION

Let A be a non-empty set. Then, $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A .

In other words, a relation I_A on a set A is called the identity relation if every element of A is related to itself only.

If $A = \{1, 2, 3, 4\}$, then $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$ is the identity relation on A whereas $\{(1, 1), (2, 2), (3, 3)\}$ and $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3)\}$ are not identity relations on A .

IV REFLEXIVE RELATION

A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for all $a \in A$.

The identity relation on a set is always reflexive. But, a reflexive relation need not be the identity relation. A relation R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Example 24 Which of the following are reflexive relations on set $A = \{1, 2, 3\}$.

$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$, $R_2 = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$

Solution R_1 is a reflexive relation on set A .

R_2 is not a reflexive relation on A because $2 \in A$ but $(2, 2) \notin R_2$

V SYMMETRIC RELATION

A relation R on a set A is said to be a symmetric relation iff $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

i.e., $aRb \Rightarrow bRa$ for all $a, b \in A$

It follows from the above definition that a relation R on a set A is symmetric iff $R = R^{-1}$. Also, the identity and universal relations on a set A are always symmetric.

Example 25 Which of the following relations is not symmetric

(A) R_1 on R defined by $(x, y) \in R_1 \Rightarrow 1 + xy > 0$ for all $x, y \in R$

(B) R_2 on $N \times N$ defined by $(a, b) R_2 (c, d)$

$\Leftrightarrow a + d = b + c$ for all $a, b, c, d \in N$

(C) R_3 on Z defined by $(a, b) \in R_3$

$\Leftrightarrow b - a$ is an even integer

(D) R_4 on power set of a set X defined by $A R_4 B$ iff $A \subseteq B$.

Ans. (D)

Solution Let $(x, y) \in R_1$. Then, $x, y \in R$ such that

$1 + xy > 0 \Rightarrow 1 + yx > 0 \Rightarrow (y, x) \in R_1$

So, R_1 is a symmetric relation on R .

We have,

$(a, b) R_2 (c, d) \Leftrightarrow a + d = b + c$

$\Leftrightarrow c + b = d + a \quad (c, d) R_2 (a, b)$

$\therefore R_2$ is a symmetric relation on $N \times N$.

Let $(a, b) \in R_3$. Then

$b - a$ is an even integer

$\Rightarrow a - b$ is an even integer

$\Rightarrow (b, a) \in R_3$

$\therefore R_3$ is a symmetric relation on Z .

Let $A, B \in P(X)$ such $A R_4 B$. Then $A \subseteq B$

This need not imply that $B \subseteq A$.

So, R_4 is not a symmetric relation on $P(X)$.

VI TRANSITIVE RELATION

A relation R on a set A is said to be a transitive relation iff

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

i.e., aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in A$.

Let S be a non-void set and R be a relation on the power set $P(S)$ defined by

$(A, B) \in R \Leftrightarrow A \subseteq B$ for all $A, B \in P(S)$

For any $A, B, C \in P(S)$, we find that

$A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$

i.e., $(A, B) \in R$ and $(B, C) \in R \Rightarrow (A, C) \in R$

So, R is a transitive relation on $P(S)$.

Example 26 Prove that on the set N of natural numbers, the relation R defined by $x R y \Rightarrow x$ is less than y is transitive.

Solution Because for any $x, y, z \in N$ $x < y$ and $y < z \Rightarrow x < z \Rightarrow x R y$ and $y R z \Rightarrow x R z$. so R is transitive.

Example 27 Show that the relation R in R defined as $R = \{(a, b) : a \leq b\}$ is transitive.

Solution Let $(a, b) \in R$, $(b, c) \in R$

$\therefore (a \leq b)$ and $b \leq c \Rightarrow a \leq c$

$\therefore (a, c) \in R$

Hence R is transitive.

VII EQUIVALENCE RELATION

A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e., $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e., $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e., $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

EQUIVALENCE CLASS

An arbitrary equivalence relation R in an arbitrary set X , R divides X into mutually disjoint subsets A_i called partitions or subdivisions of X satisfying :

- (i) All elements of A_i are related to each other, for all i .
- (ii) No element of A_i is related to any element of A_j , $i \neq j$.
- (iii) $\cup A_j = X$ and $A_i \cap A_j = \phi$, $i \neq j$.

The subsets A_i are called equivalence classes. The interesting part of the situation is that we can go reverse also.

Example 28 Let T be the set of all triangles in a plane with R a relation in T given by $R = \{(T_1, T_2) : T_1 \text{ is congruent to } T_2\}$. Show that R is an equivalence relation.

Solution R is reflexive, since every triangle is congruent to itself. Further, $(T_1, T_2) \in R \Rightarrow T_1$ is congruent to $T_2 \Rightarrow T_2$ is congruent to $T_1 \Rightarrow (T_2, T_1) \in R$. Hence, R is symmetric. Moreover, $(T_1, T_2), (T_2, T_3) \in R \Rightarrow T_1$ is congruent to T_2 and T_2 is congruent to $T_3 \Rightarrow T_1$ is congruent to $T_3 \Rightarrow (T_1, T_3) \in R$. Therefore, R is an equivalence relation.

Example 29 Which one of the following is not an equivalence relation ?

- (A) R_1 on Z defined by $a R_1 b \Leftrightarrow a - b$ is divisible by m , where m is a fixed positive integer
- (B) R_2 on R defined by $a R_2 b \Leftrightarrow 1 + ab > 0$ for all $a, b \in R$
- (C) R_3 on $N \times N$ defined by $(a, b) R_3 (c, d) \Leftrightarrow ad = bc$ for all $a, b, c, d \in N$
- (D) R_4 on Z defined by $a R_4 b \Leftrightarrow a - b$ is an even integer for all $a, b \in Z$

Ans. (B)

Solution We observe that $(1, \frac{1}{2}) \in R_2$ and $(\frac{1}{2}, -1) \in R_2$

but $(1, -1) \notin R_2$. So, R_2 is not a transitive relation. Hence, it is not an equivalence relation.

It can be easily checked that R_1, R_3 and R_4 are equivalence relations.

DPP 3

Total Marks : 30

Time 25 min

1. Question – 1 to 10 marking scheme : +3 for correct answer, -1 in all other cases. [3 × 10 = 30]

1. Let A and B be two sets such that

$$A \times B = \{(a, 1), (b, 3), (a, 3), (b, 1), (a, 2), (b, 2)\}$$

Then,

- (A) $A = \{1, 2, 3\}$ and $B = \{a, b\}$
- (B) $A = \{a, b\}$ and $B = \{1, 2, 3\}$
- (C) $A = \{1, 2, 3\}$ and $B \subset \{a, b\}$
- (D) $A \subset \{a, b\}$ and $B \subset \{1, 2, 3\}$

- 2. Let A be a non-empty set that $A \times A$ has 9 elements among which are found $(-1, 0)$ and $(0, 1)$. Then,
 - (A) $A = \{-1, 0\}$
 - (B) $A = \{0, 1\}$
 - (C) $A = \{-1, 0, 1\}$
 - (D) $A = \{-1, 1\}$
- 3. Let A and B be two non-empty sets having n elements in common. Then, the number of elements common to $A \times B$ and $B \times A$ is
 - (A) $2n$
 - (B) n
 - (C) n^2
 - (D) None of these
- 4. Let A and B be two sets having 3 elements in common. If $n(A) = 5$ and $n(B) = 4$, then $n((A \times B) \cap (B \times A)) =$
 - (A) 20
 - (B) 16
 - (C) 3
 - (D) 9
- 5. If a relation R is defined on the set Z of integers as follows : $(a, b) \in R \Leftrightarrow a^2 + b^2 = 25$. Then, Domain $(R) =$
 - (A) $\{3, 4, 5\}$
 - (B) $\{0, 3, 4, 5\}$
 - (C) $\{0, \pm 3, \pm 4, \pm 5\}$
 - (D) None of these
- 6. Let R be the relation over the set of all straight lines in a plane such that $l_1 R l_2 \Leftrightarrow l_1 \perp l_2$. Then R is
 - (A) symmetric
 - (B) reflexive
 - (C) transitive
 - (D) an equivalence relation
- 7. Let $R = \{(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)\}$ be a relation on the set $A = \{1, 2, 3, 4\}$. The relation R is
 - (A) reflexive
 - (B) transitive
 - (C) not symmetric
 - (D) a function
- 8. Let $A = \{2, 3, 4, 5, \dots, 17, 18\}$. let ' \simeq ' be the equivalence relation on $A \times A$, cartesian product of A with itself, defined by $(a, b) \simeq (c, d)$ iff $ad = bc$. Then, the number of ordered pairs of the equivalence class of $(3, 2)$ is
 - (A) 4
 - (B) 5
 - (C) 6
 - (D) 7
- 9. Let W denote the words in the English dictionary. Define the relation R by $R = \{(x, y) \in W \times W : \text{the words } x \text{ and } y \text{ have at least one letter is common}\}$. Then , R is
 - (A) not reflexive, symmetric and transitive
 - (B) reflexive, symmetric and not transitive
 - (C) reflexive, not symmetric and transitive
 - (D) reflexive, symmetric and transitive

10. The relation on the set $A = \{x : |x| < 3, x \in Z\}$ is defined by $R = \{(x, y) : y = |x|, x \neq -1\}$. Then the number of elements in the power set of R is

- (A) 32
- (B) 16
- (C) 8
- (D) 64

Result Analysis

- 1. 24 to 30 Marks : **Advance Level.**
- 2. 18 to 23 Marks : **Main Level.**
- 3. < 18 Marks : **Please go through this artical again.**

6.1 FUNCTION GENERAL DEFINATION

Definition 1 :

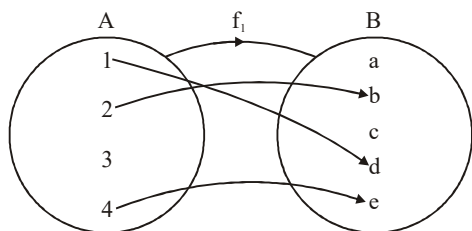
Let A and B be two non-empty sets. Then a function ‘f’ from set A to set B is a rule or method or correspondence which associates elements of set A to elements of set B such that

- (i) All elements of set A are associated to elements in set B.
- (ii) An element of set A is associated to a unique element in set B.

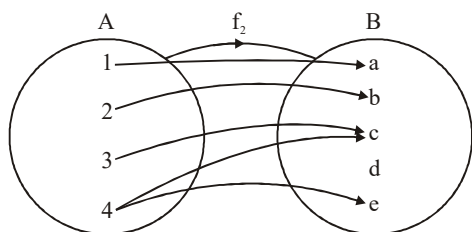
In other words, a function ‘f’ from a set A to a set B associates each element of set A to a unique element of set B. Terms such as “map” (or “mapping”), “correspondence” are used as synonyms for “functions”. If f is a function from a set A to a set B, then we write $f: A \rightarrow B$ or $A \xrightarrow{f} B$, which is read as f is a function from A to B or f maps A to B.

If an element $a \in A$ is associated to an element $b \in B$, then b is called the ‘f-image of a’ or ‘image of a under f’ or ‘the value of the function f at a’. Also, a is called the pre-image of b under the function f. We write it as $b = f(a)$ or $f: a \rightarrow b$ or $f: (a, b)$

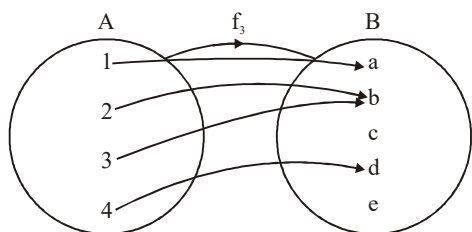
e.g., Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$ be two sets and let f_1, f_2, f_3 and f_4 be rules associating elements (A to elements of) B as show in the following figures.



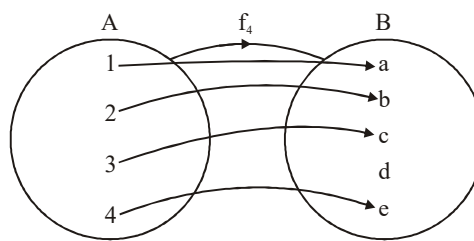
1. Not a Function



2. Not a Function



3. Function



4. Function

6.2 Function as a Set of Ordered Pairs

A function $f: A \rightarrow B$ can be expressed as a set of ordered pairs in which each ordered pair is such that its first element belongs to A and the second element is the corresponding element of B.

As such a function $f: A \rightarrow B$ can be considered as a set of ordered pairs $(a, f(a))$, where $a \in A$ and $f(a) \in B$, which is the f image of a. Hence, f is a subset of $A \times B$. As a particular type of relation, we can define a function as follows :

Definition 2 :

A relation R from a set A to a set B is called a function if

- (i) Each element of A is associated with some element of B and
 - (ii) Each element of A has a unique image in B.
- Thus, a function f from a set A to a set B is a subset of $A \times B$ in which each $a \in A$ appears in one and only one ordered pair belonging to f. Hence, a function f is a relation from A to B satisfying the following properties :
- (i) $f \subset A \times B$
 - (ii) $\forall a \in A, \text{ or } (a, f(a)) \in f$
 - (iii) $(a, b) \in f \text{ and } (a, c) \in f \Rightarrow b = c$

Thus, the ordered pair of f must satisfy the property that each element of A appears in some ordered pair and no two ordered pairs have the same first element.

NOTE :

Every function is a relation but every relation is not necessarily a function.

Example 30 Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ be two sets.

Which one of the following subsets of $A \times B$ defines a function from A to B ?

- (a) $f_1 = \{(1, 2), (2, 3), (3, 4)\}$
- (b) $f_2 = \{(1, 2), (1, 3), (2, 3), (3, 4)\}$
- (c) $f_3 = \{(1, 3), (2, 4)\}$
- (d) $f_4 = \{(1, 4), (2, 4), (3, 4), (2, 3)\}$

Solution : (a)

We observe that corresponding to each element in A there is unique ordred pair in f_1 . So f_1 is a function from A to B. f_2 is not a function, because there are two ordered pairs (1, 2) and (1, 3), in f_3 corresponding to $1 \in A$.

As there is no ordered pair in f_3 corresponding to $3 \in A$. So, it is not a function from A to B.

Similarly, f_4 is not a function from A to B as there are two ordered pairs (2, 3) and (2, 4) in f_4 corresponding to $2 \in A$.

Example 31 If $A = \{1, 2, 3, 4\}$, then which of the following are functions from A to itself ?

- (a) $f_1 = \{(x, y) : y = x + 1\}$ (b) $f_2 = \{(x, y) : x + y \geq 4\}$
 (c) $f_3 = \{(x, y) : y < x\}$ (d) $f_4 = \{(x, y) : x + y = 5\}$

Solution : (d)

We have,

$$f_1 = \{(1, 2), (2, 3), (3, 4)\},$$

$$f_2 = \{(1, 4), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4), (4, 1), (4, 2), (4, 3)\}$$

$$f_3 = \{(2, 1), (3, 1), (3, 2), \dots\} \text{ and}$$

$$f_4 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}.$$

Obviously, f_4 is a function from A to itself and f_1, f_2, f_3 are not functions from A to itself.

6.3 NUMBER OF FUNCTIONS

Let A and B be two finite sets having m and n elements respectively. Then, each element of set A can be associated to any one of n elements of set A. So, total number of functions from set A to set B is equal to the number of ways of doing m jobs where each job can be done in n ways. The total number of such ways is

$$n \times n \times n \dots \times n = n^m.$$

m-times

Hence, the total number of functions from A to B is n^m i.e., $[O(B)]^{O(A)}$.

$$O(A) = \text{No. of element in set A} = m$$

$$O(B) = \text{No. of element in set B} = n$$

For example, the total number of functions from a set

$$A = \{a, b, c, d\} \text{ to a set } B = \{1, 2, 3\} \text{ is } 3^4 = 81.$$

The total number of relations from a set A having m elements to a set n having n elements is 2^{mn} . So, the number of relations from A to B which are not functions is $2^{mn} - n^m$ i.e., $2^{O(A) \times O(B)} - [O(B)]^{O(A)}$.

6.4 REAL FUNCTION

If the domain and co-domain of a function are subsets of R (Set of all real numbers). It is called a real valued function or in short a real function.

Description Of A Real Function

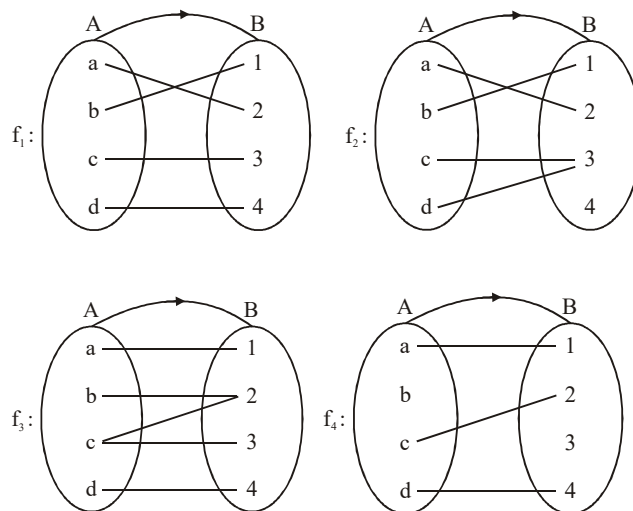
If f is a real valued function with finite domain, then f can be described by listing the values which it attains at different points of its domain. However, if the domain of a real function is an infinite set, then, f cannot be described by listing the values at points in its domain. In such cases real functions are generally described by some general formula or rule like $f(x) = x^2 + 1$ or $f(x) = 2 \sin x + 3$ etc. In calculus almost all real functions are described by some general formula or rule.

6.5 REPRESENTATION OF FUNCTION

It can be done by three methods :

- (A) By Mapping
 (B) By Algebraic Method
 (C) In the form of Ordered pairs

(A) Mapping : It shows the graphical aspect of the relation of the elements of A with the elements of B



In the above given mappings rule f_1 and f_2 shows a function because each element of A is associated with a unique element of B. Whereas f_3 and f_4 are not function because in f_3 , element c is associated with two elements of B, and in f_4 , b is not associated with any element of B, which do not follow the definition of function. In f_2 , c and d are associated with same element, still it obeys the rule of definition of function because it does not tell that every element of A should be associated with different elements of B.

(B) Algebraic Method : It shows the relation between the elements of two sets in the form of two variables x and y where x is independent variable and y is dependent variable.

If A and B be two given sets

$$A = \{1, 2, 3\}, B = \{5, 7, 9\} \text{ then } f : A \rightarrow B, y = f(x) = 2x + 3.$$

(C) In the form of ordered pairs : A function $f : A \rightarrow B$ can be expressed as a set of ordered pairs in which first element of every ordered pair is a member of A and second element is the member of B. So f is a set of ordered Pairs (a, b) such that :

- (i) a is an element of A
- (ii) b is an element of B
- (iii) Two ordered pairs should not have the same first element.

Example 32 If A and B are two sets such that $A = \{1, 2, 3\}$ and $B = \{5, 7, 9\}$ and if a function is defined from $f : A \rightarrow B$, $f(x) = 2x + 3$ then find a function in the form of ordered pairs.

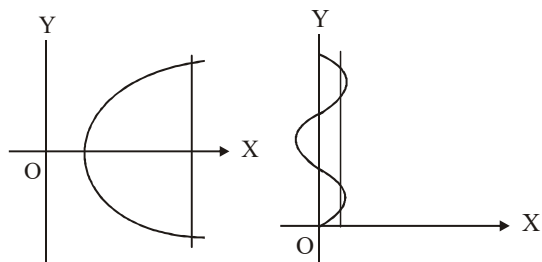
Solution : $f : A \rightarrow B$, $f(x) = 2x + 3$
 $f(1) = 2 \cdot 1 + 3 = 5$, $f(2) = 2 \cdot 2 + 3 = 7$

$$f(3) = 2 \cdot 3 + 3 = 9 \quad \therefore \quad f: \{(1,5), (2,7), (3,9)\}$$

6.6 TESTING FOR A FUNCTION

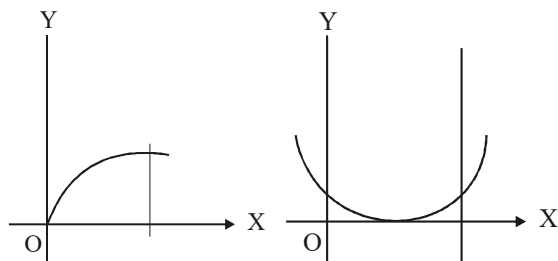
Vertical Line Test :

If we are given a graph of the relation then we can check whether the given relation is function or not . If it is possible to draw a vertical line which cuts the given curve at more than one point then given relation is not a function and when this vertical line means line parallel to Y-axis cuts the curve at only one point then it is a function.



(i)

(ii)



(iii)

(iv)

fig (iii) and (iv) represents a function.

6.7 DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION :

Let $f : A \rightarrow B$, then the set A is known as the domain of f & the set B is known as co-domain of f . The set of f images of all the elements of A is known as the range of f .

Thus : Domain of $f = \{a \mid a \in A, (a, f(a)) \in f\}$

Range of $f = \{f(a) \mid a \in A, f(a) \in B\}$

NOTE :

- It should be noted that range is a subset of co-domain.
- If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range.

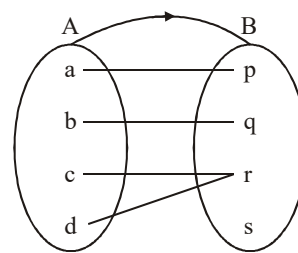
Domain = All possible values of x for which $f(x)$ exists.

Range = For all values of x , all possible values of $f(x)$.

Domain = $\{a, b, c, d\} = A$

Co-domain = $\{p, q, r, s\} = B$

Range = $\{p, q, r\}$



Domain And Range Of A Real Function

Domain :

The domain is the set of all real numbers x for which $f(x)$ is a real number.

Range :

Range of a real function f is the set of all points y such that $y = f(x)$, where $x \in \text{Dom } f(x)$.

The following algorithm is very helpful in determining the range of real functions.

6.8 RULE TO CALCULATE DOMAIN OF FUNCTION :

Let f and g be two given functions and their domain are D_f and D_g respectively, then the sum, difference, product and quotient functions are defined as :

$$(a) \quad (f + g)(x) = f(x) + g(x), \quad \forall x \in D_f \cap D_g$$

$$(b) \quad (f - g)(x) = f(x) - g(x), \quad \forall x \in D_f \cap D_g$$

$$(c) \quad (f \cdot g)(x) = f(x) \cdot g(x), \quad \forall x \in D_f \cap D_g$$

$$(d) \quad (f/g)(x) = \frac{f(x)}{g(x)}; \quad g(x) \neq 0, \quad \forall x \in D_f \cap D_g$$

Example 33 Find the domain of the following function

$$(i) \quad f(x) = \sqrt{1-x} \qquad (ii) \quad f(x) = \frac{1}{2x+1}$$

$$(iii) \quad f(x) = \sqrt{4x+20} - \frac{1}{x-1}$$

Solution (i) $f(x) = \sqrt{1-x}$

$$\text{We know that } 1-x \geq 0 \quad \Rightarrow \quad x-1 \leq 0 \quad \Rightarrow \quad x \leq 1$$

So, domain of $f(x) = (-\infty, 1]$

$$(ii) \quad f(x) = \frac{1}{2x+1}$$

$$\text{We know that } 2x+1 \neq 0 \quad \Rightarrow \quad x \neq -\frac{1}{2}$$

So, domain of $f(x) = \mathbb{R} - \{-\frac{1}{2}\}$

$$(iii) \quad f(x) = \sqrt{4x+20} - \frac{1}{x-1}$$

$$\text{Let } g(x) = \sqrt{4x+20} \quad \text{and} \quad h(x) = \frac{1}{x-1}$$

$$\text{Domain of } g(x) \text{ be } 4x+20 \geq 0 \quad \Rightarrow \quad x \geq -5 \Rightarrow x \in [-5, \infty)$$

$$\text{Domain of } h(x) \text{ be } x-1 \neq 0 \quad \Rightarrow \quad x \in \mathbb{R} - \{1\}$$

Domain of $f(x) = \text{Domain of } g(x) \cap \text{Domain of } h(x)$.

Domain of $f(x) = [-5, \infty) \cap \mathbb{R} - \{1\}$

Domain of $f(x) = [-5, \infty) - \{1\}$

Method to Find Range of Real function

STEP 1 Put $f(x) = y$

STEP 2 Solve the equation in step 1 for x to obtain $x = \phi(y)$.

STEP 3 Find the values of y for which the values of x , obtained from $x = \phi(y)$ are in the domain of f .

STEP 4 The set of values of y obtained in step-3 is the range of f .
If range of $f(x)$ is known then we find the range of $f(x)$, $|f(x)|$,

$\sqrt{f(x)}$, $\frac{1}{f(x)}$, $(f(x))^2$ with the help of following table.

$f(x)$	$ f(x) $	$\sqrt{f(x)}$	$\frac{1}{f(x)}$	$(f(x))^2$
$[1, 4]$	$[1, 4]$	$[1, 2]$	$\left[\frac{1}{4}, 1\right]$	$[1, 16]$
$[-9, -4]$	$[4, 9]$	ϕ	$\left[-\frac{1}{4}, -\frac{1}{9}\right]$	$[16, 81]$
$[-4, 9]$	$[4, 9]$	$[0, 3]$	$\left(-\infty, \frac{1}{4}\right] \cup \left[\frac{1}{9}, \infty\right)$	$[0, 81]$

Example 34 Find the range of following function

(i) $f(x) = \frac{1}{x}$

(ii) $f(x) = \ln x$

(iii) $f(x) = x^2$

(iv) $f(x) = \frac{x^2}{1+x^2}$

Solution (i) $f(x) = \frac{1}{x}$, replace $f(x)$ by y .

We get $y = \frac{1}{x}$

Calculate x in term of y . i.e., $x = \frac{1}{y}$

Set of all values of y is the range of function.

Here $y \neq 0, \forall y \in \mathbb{R} - \{0\}$ x is defined

So, range = $\mathbb{R} - \{0\}$

(ii) $f(x) = \ln x$

$y = \ln x \Rightarrow x = e^y$

Here $\forall y \in \mathbb{R}$ x is defined

So range = \mathbb{R}

(iii) $f(x) = x^2$

$y = x^2 \Rightarrow x = \sqrt{y}$

Here $y \geq 0, \forall y \geq 0$ x is real value

So range = $[0, \infty)$

(iv) $f(x) = \frac{x^2}{1+x^2}$

$y = \frac{x^2}{1+x^2} \Rightarrow y + x^2y = x^2 \Rightarrow x^2(1-y) = y$

$x^2 = \frac{y}{1-y} \Rightarrow x = \sqrt{\frac{y}{1-y}}$

$\frac{y}{1-y} \geq 0$ or $y \leq 0$ & $1-y < 0$

$y \geq 0$ & $1-y > 0$ $y > 1$

$y \in [0, 1)$ $y \in \phi$

So $\forall y \in [0, 1)$ x is real.

Hence, range = $[0, 1)$

Example 35 Find the domain and range of

$f(x) = \sqrt{x^2 - 3x + 2}$

Solution : For domain,

$x^2 - 3x + 2 \geq 0$

or $(x-1)(x-2) \geq 0$

or $x \in (-\infty, 1] \cup [2, \infty)$

Now, $f(x) = \sqrt{x^2 - 3x + 2}$

$= \sqrt{\left(x - \frac{3}{2}\right)^2 + 2 - \frac{9}{4}}$

$= \sqrt{\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}}$

Now, the least permissible value of $\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$ is 0 when

$\left(x - \frac{3}{2}\right)^2 = \pm \frac{1}{2}$

Hence, the range is $[0, \infty)$

Example 36 Find domain of $f(x) = \frac{x-3}{(x+3)\sqrt{x^2-4}}$

Solution $f(x) = \frac{x-3}{(x+3)\sqrt{x^2-4}}$

We must have $x^2 - 4 > 0$ and $x + 3 \neq 0$

$(x+2)(x-2) > 0$ and $x \neq -3$

$x \in (-\infty, -2) \cup (2, \infty)$ and $x \neq -3$

Required answer $x \in (-\infty, -3) \cup (-3, -2) \cup (2, \infty)$

Example 37 Find the range of the function

$f(x) = 6^x + 3^x + 6^{-x} + 3^{-x} + 2$.

Solution : $f(x) = 6^x + 3^x + 6^{-x} + 3^{-x} + 2$.

$$= (\sqrt{6^x} - \sqrt{6^{-x}})^2 + (\sqrt{3^x} - \sqrt{3^{-x}})^2 + 6 \geq 6$$

$$\therefore (\sqrt{6^x} - \sqrt{6^{-x}})^2 + (\sqrt{3^x} - \sqrt{3^{-x}})^2 \geq 0, \forall x \in \mathbb{R}$$

Hence, the range is $[6, \infty)$

Example 38 Find the range of $f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$

Solution Let $y = \frac{x^2 - x + 1}{x^2 + x + 1}$

or $(1 - y)x^2 - (1 + y)x + 1 - y = 0$

Now, x is real.

Then, $D \geq 0$

or $(1 + y)^2 - 4(1 - y)^2 \geq 0$

or $(1 + y - 2 + 2y)(1 + y + 2 - 2y) \geq 0$

or $(3y - 1)(3 - y) \geq 0$

or $3\left(y - \frac{1}{3}\right)(y - 3) \leq 0$ or $\frac{1}{3} \leq y \leq 3$

Hence, the range is $\left[\frac{1}{3}, 3\right]$.

Example 39 Find the range of $f(x) = \frac{x^2 + 1}{x^2 + 2}$

Solution $f(x) = \frac{x^2 + 1}{x^2 + 2} = \frac{x^2 + 2 - 1}{x^2 + 2} = 1 - \frac{1}{x^2 + 2}$

Now $x^2 + 2 \geq 2, \forall x \in \mathbb{R}$

$$\Rightarrow 0 < \frac{1}{x^2 + 2} \leq \frac{1}{2} \Rightarrow -\frac{1}{2} \leq -\frac{1}{x^2 + 2} < 0$$

$$\Rightarrow \frac{1}{2} \leq 1 - \frac{1}{x^2 + 2} < 1$$

Example 40 Find the range of $f(x) = \sqrt{x-1} + \sqrt{5-x}$

Solution Let $y = \sqrt{x-1} + \sqrt{5-x}$

or $y^2 = x - 1 + 5 - x + 2\sqrt{(x-1)(5-x)}$

or $y^2 = 4 + 2\sqrt{-x^2 - 5 + 6x}$

or $y^2 = 4 + 2\sqrt{4 - (x-3)^2}$

Then y^2 has minimum value of 4 [when $4 - (x-3)^2 = 0$] and maximum value 8 when $x = 3$.

Therefore, $y \in [2, 2\sqrt{2}]$.

Example 41 Find range of value of x for which

$$f(x) = \sqrt{2-x} - \frac{1}{\sqrt{9-x^2}} \text{ is defined.}$$

Solution $f(x) = \sqrt{2-x} - \frac{1}{\sqrt{9-x^2}}$

We must have $2-x \geq 0$ and $9-x^2 > 0$

$$x \leq 2 \quad \text{and} \quad x^2 < 9$$

$$x \leq 2 \quad \text{and} \quad -3 < x < 3$$

Required answer $x \in (-3, 2]$

Example 42 Find domain of $f(x) = \sqrt{x - \sqrt{1-x^2}}$

Solution $f(x) = \sqrt{x - \sqrt{1-x^2}}$ to get defined $x - \sqrt{1-x^2} \geq 0$
and $1-x^2 \geq 0$

$$\Rightarrow x - \sqrt{1-x^2} \geq 0$$

$$\Rightarrow x \geq \sqrt{1-x^2}$$

As x has to be positive, so we can square this inequality

$$\Rightarrow x^2 \geq 1-x^2$$

$$\Rightarrow x^2 \geq \frac{1}{2}$$

$$\Rightarrow x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, \infty\right) \quad \dots (i)$$

$$\Rightarrow 1-x^2 \geq 0$$

$$\Rightarrow x^2 \leq 1$$

$$\Rightarrow x \in [-1, 1] \quad \dots (ii)$$

from (i) and (ii), $x \in \left[\frac{1}{\sqrt{2}}, 1\right]$. As ' x ' has to be positive.

Example 43 Find the range of the function

$$f(x) = 2|\sin x| - 3|\cos x|.$$

Solution $f(x) = 2|\sin x| - 3|\cos x|.$

We know that subtraction of two continuous function is always continuous.

So, here $f(x)$ is continuous function.

$f(x)$ has minimum value when $x = 0$

So $f_{\min} = -3$ at $x = 0$ and

$f(x)$ has maximum value when $x = \pi/2$

So $f_{\max} = 2$ at $x = \pi/2$

Hence range = $[-3, 2]$

Example 44 Find the range of function

$$f(x) = \sqrt{\sin(\cos x)} + \sqrt{\cos(\sin x)}$$

Solution $f(x) = \sqrt{\sin(\cos x)} + \sqrt{\cos(\sin x)}$

It's fundamental period is 2π .

For range take on interval of 2π length

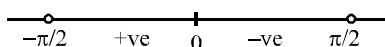
Suppose it is $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

But for $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $f(x)$ is not real

Now we concentrate for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$f'(x) = \frac{\cos(\cos x) \cdot (-\sin x)}{2\sqrt{\sin(\cos x)}} - \frac{\sin(\sin x) \cdot \cos x}{2\sqrt{\cos(\sin x)}} = 0$$

Since, $x = 0$ satisfies the above equation



Signs scheme for $f'(x)$

$$f_{\min} = f\left(\frac{\pi}{2}\right) = f\left(-\frac{\pi}{2}\right) = \sqrt{\cos 1}$$

$$f_{\max} = f(0) = \sqrt{\sin(\cos 0)} + \sqrt{\cos(\sin 0)} = \sqrt{\sin 1} + 1$$

$$\therefore R_f = \left[\sqrt{\cos 1}, 1 + \sqrt{\sin 1}\right]$$

DPP 4

Total Marks 30

Time 20 Minute

- Question Number 1 to 4. **Marking Scheme** : +3 for correct answer -1 in all other cases. [(4.3=12)]
- Question Number 5 to 10. **Marking Scheme** : +3 for correct answer 0 in all other cases. [(6.3=18)]

- Find the range of $f(x) = x^2 - x - 3$.
- Find the domain and range of $f(x) = \sqrt{x^2 - 4x + 6}$
- Find the range of $f(x) = \frac{x^2 + 34x - 71}{x^2 + 2x - 7}$
- Find the domain and range of $f(x) = \sqrt{3 - 2x - x^2}$.
- If $a, b \in \{1, 2, 3, 4\}$, then which of the following are functions in the given set ?
 - $f_1 = \{(x, y) : y = x + 1\}$
 - $f_2 = \{(x, y) : x + y > 4\}$
 - $f_3 = \{(x, y) : y < x\}$
 - $f_4 = \{(x, y) : x + y = 5\}$
- Given $A = \{-1, 0, 2, 5, 6, 11\}$, $B = \{-2, -1, 0, 18, 28, 108\}$ and $f(x) = x^2 - x - 2$. Is $f(A) = B$? Find $f(A)$.

7. Find the domain of $f(x) = \sqrt{\frac{x-2}{x+2}} + \sqrt{\frac{1-x}{1+x}}$

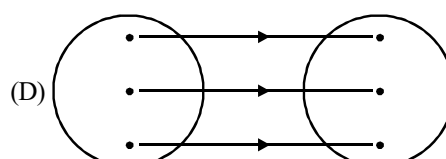
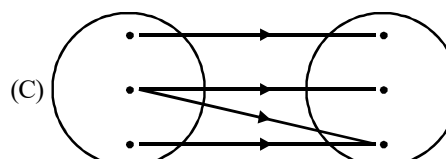
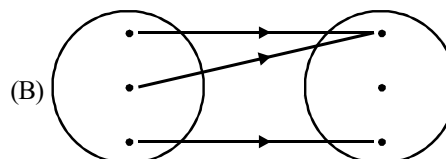
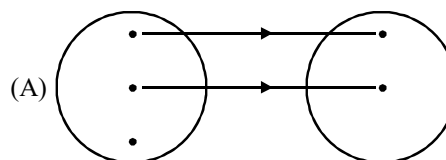
8. Which among the following relations is a function ?

- (A) $x^2 + y^2 = r^2$ (B) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$
 (C) $y^2 = 4ax$ (D) $x^2 = 4ay$

9. Which of the following correspondences can be called a function ?

- (A) $f(x) = x^3$; $\{-1, 0, 1\} \rightarrow \{0, 1, 2, 3\}$
 (B) $f(x) = \pm\sqrt{x}$; $\{0, 1, 4\} \rightarrow \{-2, -1, 0, 1, 2\}$
 (C) $f(x) = \sqrt{x}$; $\{0, 1, 4\} \rightarrow \{-2, -1, 0, 1, 2\}$
 (D) $f(x) = -\sqrt{x}$; $\{0, 1, 4\} \rightarrow \{-2, -1, 0, 1, 2\}$

10. Which of the following pictorial diagrams represent the function



Result Analysis

- 24 to 30 Marks : **Advanced Level.**
- 15 to 23 Marks : **Mains Level.**
- < 15 Marks : **Please go through this artical again.**

7 TYPES OF FUNCTIONS

7.1 Quadratic Function

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. We have

$$\begin{aligned}
 f(x) &= a \left[x^2 + \frac{b}{a}x + \frac{c}{a} \right] \\
 &= a \left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2} \right] \\
 &= a \left[\left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right] \\
 &= a \left[\left(x + \frac{b}{2a} \right)^2 - \frac{D}{4a^2} \right]
 \end{aligned}$$

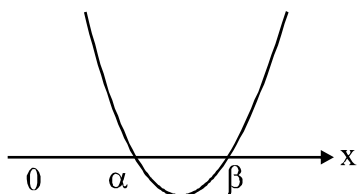
or $\left(y + \frac{D}{4a} \right) = a \left(x + \frac{b}{2a} \right)^2$

Thus, $y = f(x)$ represents a parabola whose axis is parallel to the y-axis and vertex is $A \left(-\frac{b}{2a}, -\frac{D}{4a} \right)$. For some

values of x , $f(x)$ may be positive, negative, or zero. For $a > 0$, the parabola opens upwards and for $a < 0$, the parabola opens downwards.

This gives the following cases :

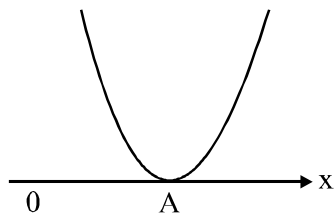
- $a > 0$ and $D > 0$
Let $f(x) = 0$ has two real roots α and β (where $\alpha < \beta$). Then,



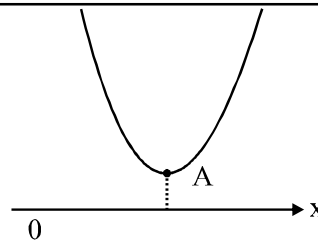
$$f(x) > 0 \quad \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$$

$$\text{and } f(x) < 0 \quad \forall x \in (\alpha, \beta)$$

- $a > 0$ and $D = 0$
So, $f(x) \geq 0 \quad \forall x \in \mathbb{R}$, i.e., $f(x)$ is positive for all values of x except at the vertex where $f(x) = 0$.



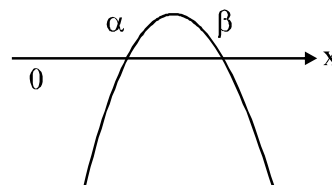
- $a > 0$ and $D < 0$
So, $f(x) > 0 \quad \forall x \in \mathbb{R}$. i.e., $f(x)$ is positive for all values of x .



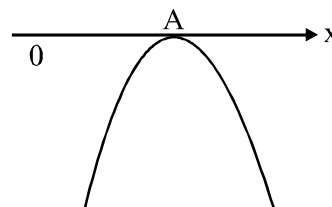
The range of function is $\left[-\frac{D}{4a}, \infty \right)$.

$x = -\frac{b}{2a}$ is point of minima.

- $a < 0$ and $D > 0$
Let $f(x) = 0$ has two roots α and β (where $\alpha < \beta$). Then,
 $f(x) < 0 \quad \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$
and $f(x) > 0 \quad \forall x \in (\alpha, \beta)$



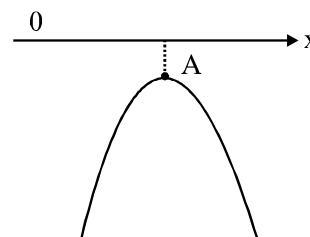
- $a < 0$ and $D = 0$
So, $f(x) \leq 0 \quad \forall x \in \mathbb{R}$, i.e., $f(x)$ is negative for all values of x except at the vertex where $f(x) = 0$.



- $a < 0$ and $D < 0$
So, $f(x) < 0 \quad \forall x \in \mathbb{R}$, i.e., $f(x)$ is negative for all values of x

The range of function is $\left(-\infty, -\frac{D}{4a} \right]$

$x = -\frac{b}{2a}$ is point of maxima.



NOTE :

- If $f(x) \geq 0 \forall x \in \mathbb{R}$, then $a > 0$ and $D \leq 0$
- If $f(x) \leq 0 \forall x \in \mathbb{R}$, then $a < 0$ and $D \leq 0$

Example 45 If $f(x) = \sqrt{x^2 + ax + 4}$ is defined for all x , then find the values of a .

Solution $f(x) = \sqrt{x^2 + ax + 4}$ is defined for all x . Therefore,
 $x^2 + ax + 4 \geq 0$ for all x
 or $D = a^2 - 16 \leq 0$
 or $a \in [-4, 4]$

Example 46 Find the complete set of values of a such that

$\frac{x^2 - x}{1 - ax}$ attains all real values.

Solution $y = \frac{x^2 - x}{1 - ax}$

or $x^2 - x = y - axy$
 or $x^2 + x(ay - 1) - y = 0$

Since x is real we get

$$(ay - 1)^2 + 4y \geq 0$$

or $a^2y^2 + 2y(2 - a) + 1 \geq 0 \forall y \in \mathbb{R}$

So, as $a^2 > 0$

$$4(2 - a)^2 - 4a^2 \leq 0$$

or $4 - 4a \leq 0$

or $a \in [1, \infty)$

Now at $a = 1$, $y = \frac{x^2 - x}{1 - x}$

$$y = \frac{x(1 - x)}{(1 - x)}, \quad (\text{common factor})$$

$\Rightarrow y = x, \quad x \neq 1$

\Rightarrow if $x \neq 1 \Rightarrow y \neq 1 \Rightarrow y \notin \mathbb{R}$

So, $a \neq 1$ then $a \in (1, \infty)$

7.2 TRIGONOMETRIC FUNCTIONS

1. $y = f(x) = \sin x$

Domain : \mathbb{R}

Range : $[-1, 1]$

$\sin^2 x, |\sin x| \in [0, 1]$

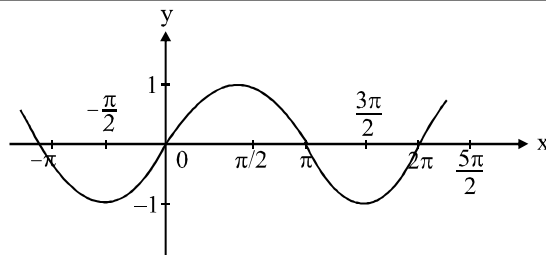
$\sin x = 0 \Rightarrow x = n\pi, n \in \mathbb{I}$

$\sin x = 1 \Rightarrow x = (4n + 1)\pi/2, n \in \mathbb{I}$

$\sin x = -1 \Rightarrow x = (4n - 1)\pi/2, n \in \mathbb{I}$

$\sin x = \sin \alpha \Rightarrow x = n\pi + (-1)^n \alpha, n \in \mathbb{I}$

$\sin x \geq 0 \Rightarrow x \in \bigcup_{n \in \mathbb{I}} [2n\pi, \pi + 2n\pi]$



2. $y = f(x) = \cos x$

Domain : \mathbb{R}

Range : $[-1, 1]$

$\cos^2 x, |\cos x| \in [0, 1]$

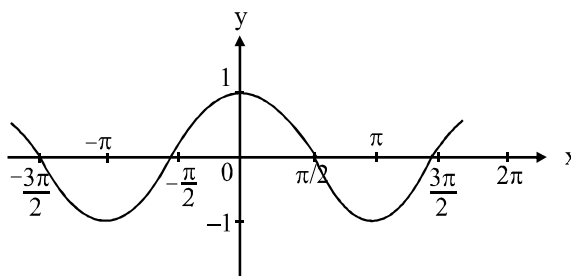
$\cos x = 0 \Rightarrow x = (2n + 1)\pi/2, n \in \mathbb{I}$

$\cos x = 1 \Rightarrow x = 2n\pi, n \in \mathbb{I}$

$\cos x = -1 \Rightarrow x = (2n + 1)\pi, n \in \mathbb{I}$

$\cos x = \cos \alpha \Rightarrow x = 2n\pi \pm \alpha, n \in \mathbb{I}$

$\cos x \geq 0 \Rightarrow x \in \bigcup_{n \in \mathbb{I}} \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2} \right]$



3. $y = f(x) = \tan x$

Domain : $\mathbb{R} \sim (2n + 1)\pi/2, n \in \mathbb{I}$

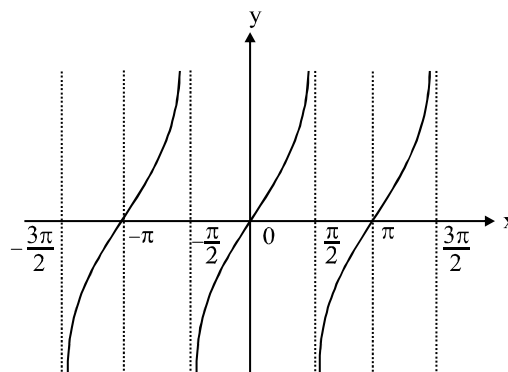
Range : $(-\infty, \infty)$

Discontinuous at $x = (2n + 1)\pi/2, n \in \mathbb{I}$

$\tan^2 x, |\tan x| \in [0, \infty)$

$\tan x = 0 \Rightarrow x = n\pi, n \in \mathbb{I}$

$\tan x = \tan \alpha \Rightarrow x = n\pi + \alpha, n \in \mathbb{I}$



4. $y = f(x) = \cot x$

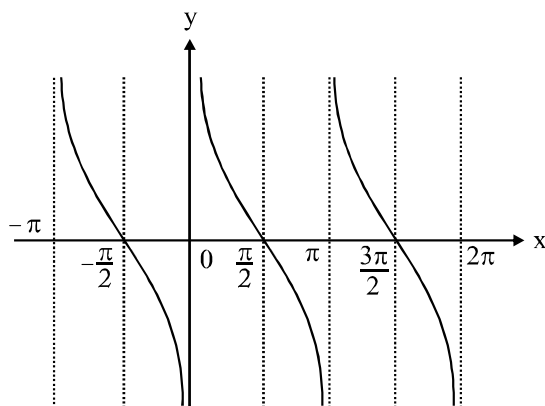
Domain : $\mathbb{R} \sim n\pi, n \in \mathbb{I}$

Range : $(-\infty, \infty)$

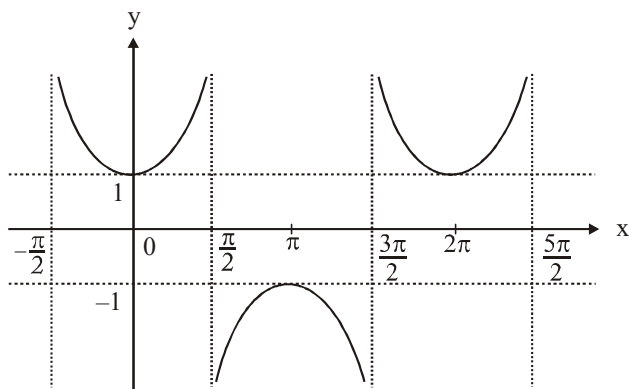
Discontinuous at $x = n\pi, n \in \mathbb{I}$

$$\cot^2 x, |\cot x| \in [0, \infty)$$

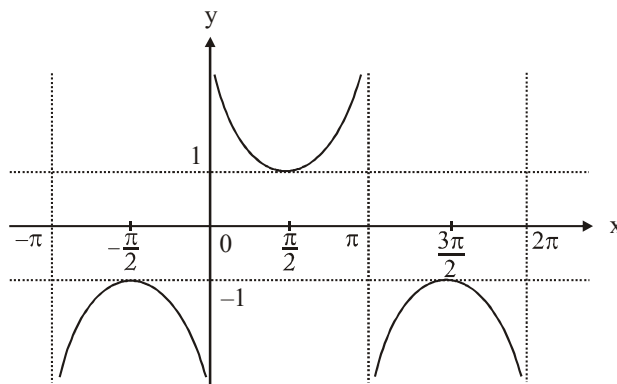
$$\cot x = 0 \Rightarrow x = (2n+1)\pi/2, n \in \mathbb{I}$$



5. $y = f(x) = \sec x$
 Domain : $\mathbb{R} \sim (2n+1)\pi/2, n \in \mathbb{I}$
 Range : $(-\infty, -1] \cup [1, \infty)$
 $\sec^2 x, |\sec x| \in [1, \infty)$
 actual domain



6. $y = f(x) = \operatorname{cosec} x$
 Domain : $\mathbb{R} \sim n\pi, n \in \mathbb{I}$
 Range : $(-\infty, -1] \cup [1, \infty)$
 $\operatorname{cosec}^2 x, |\operatorname{cosec} x| \in [1, \infty)$



Note : $f(x) = a \cos x + b \sin x = \sqrt{a^2 + b^2} \sin\left(x + \tan^{-1} \frac{a}{b}\right)$

$$= \sqrt{a^2 + b^2} \cos\left(x - \tan^{-1} \frac{b}{a}\right)$$

Proof : Let $a = r \sin \alpha, b = r \cos \alpha$, Then,

$$a^2 + b^2 = r^2 \text{ and } \tan \alpha = \frac{a}{b}$$

$$\text{Now, } f(x) = r (\cos x \sin \alpha + \sin x \cos \alpha)$$

$$= r \sin(x + \alpha) = \sqrt{a^2 + b^2} \sin\left(x + \tan^{-1} \frac{a}{b}\right)$$

$$\text{Since } -1 \leq \sin\left(x + \tan^{-1} \frac{a}{b}\right) \leq 1$$

we have

$$-\sqrt{a^2 + b^2} \leq \sqrt{a^2 + b^2} \sin\left(x + \tan^{-1} \frac{a}{b}\right) \leq \sqrt{a^2 + b^2}$$

So, the range of $f(x) = a \cos x + b \sin x$ is

$$\left[-\sqrt{a^2 + b^2}, \sqrt{a^2 + b^2}\right].$$

Example 47 Find the domain of the function

$$f(x) = \frac{1}{1 + 2 \sin x}$$

Solution To define $f(x)$, we must have $1 + 2 \sin x \neq 0$

$$\text{or } \sin x \neq -\frac{1}{2} \text{ or } x \neq n\pi + (-1)^n \frac{7\pi}{6}, n \in \mathbb{Z}$$

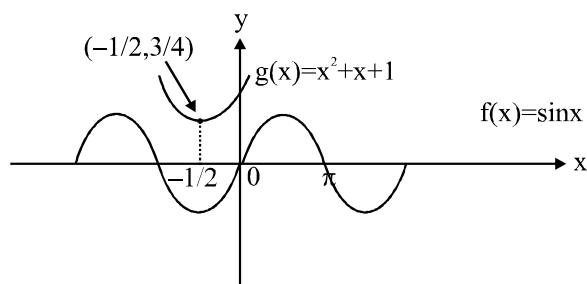
Hence, the domain of the function is

$$\mathbb{R} - \left\{ n\pi + (-1)^n \frac{7\pi}{6}, n \in \mathbb{Z} \right\}$$

Example 48 Find the number of solutions of the equation $\sin x = x^2 + x + 1$.

Solution Let $f(x) = \sin x$ and $g(x) = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

as shown in the figure, which do not intersect at any point. Therefore, there is no solution.



Example 49 Find the range of $f(x) = \sin^2 x - \sin x + 1$.

Solution $f(x) = \sin^2 x - \sin x + 1 = \left(\sin x - \frac{1}{2}\right)^2 + \frac{3}{4}$

Now, $-1 \leq \sin x \leq 1$ or $-\frac{3}{2} \leq \sin x - \frac{1}{2} \leq \frac{1}{2}$

or $0 \leq \left(\sin x - \frac{1}{2}\right)^2 \leq \frac{9}{4}$ or $\frac{3}{4} \leq \left(\sin x - \frac{1}{2}\right)^2 + \frac{3}{4} \leq 3$

Hence, the range is $\left[\frac{3}{4}, 3\right]$

Example 50 Find the range of

$$f(\theta) = 5 \cos \theta + 3 \cos \left(\theta + \frac{\pi}{3}\right) + 3$$

Solution $f(\theta) = 5 \cos \theta + 3 \cos \left(\theta + \frac{\pi}{3}\right) + 3$

$$= 5 \cos \theta + \frac{3}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta + 3$$

$$= \frac{13}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta + 3$$

$$= \sqrt{\left(\frac{169}{4} + \frac{27}{4}\right)} \sin(\theta - \alpha) + 3$$

Thus, the range of $f(\theta)$ is $[-4, 10]$.

Example 51 Find the domain of $f(x) = \sqrt{\cos(\sin x)}$

Solution $f(x) = \sqrt{\cos(\sin x)}$ is defined if

$$\cos(\sin x) \geq 0$$

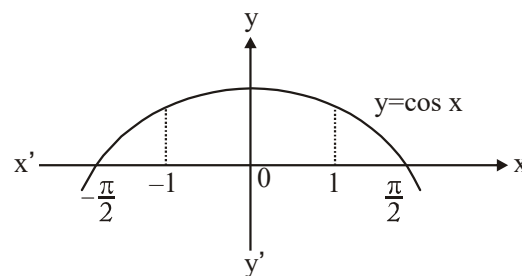
As we know, $-1 \leq \sin x \leq 1$ for all x . So, $\cos \theta \geq 0$

(here, $\theta = \sin x$ lies in the first and fourth quadrants)

i.e., $\cos(\sin x) \geq 0$ for all x

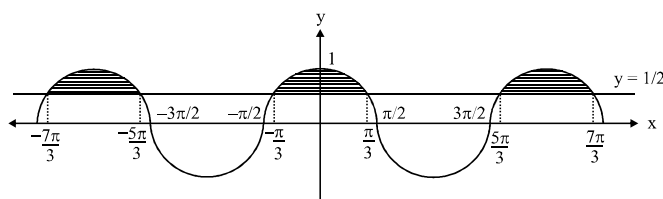
i.e., $x \in \mathbb{R}$

Thus, the domain $f(x)$ is \mathbb{R} .



Example 52 Find the domain of $\sqrt{\cos x - \frac{1}{2}}$.

Solution Draw graph of $y = \cos x$ and $y = \frac{1}{2}$



The shaded region is the required answer

So,

$$x \in \left\{ \dots \left[\frac{-7\pi}{3}, \frac{-5\pi}{3} \right] \cup \left[\frac{-\pi}{3}, \frac{\pi}{3} \right] \cup \left[\frac{5\pi}{3}, \frac{7\pi}{3} \right] \dots \right\}$$

general form $x \in \left[2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3} \right], n \in \mathbb{I}$

Example 53 If $f(x) = \frac{\sin x}{\sqrt{1 + \tan^2 x}} - \frac{\cos x}{\sqrt{1 + \cot^2 x}}$, then find the

range of $f(x)$.

Solution

$$f(x) = \frac{\sin x}{|\sec x|} - \frac{\cos x}{|\cos x|} = \sin x |\cos x| - \cos x |\sin x|$$

Clearly, the domain of $f(x)$ is $\mathbb{R} \sim \left\{ n\pi, (2n+1)\frac{\pi}{2} / n \in \mathbb{I} \right\}$

$$\text{and } f(x) = \begin{cases} 0, & x \in (0, \pi/2) \\ -\sin 2x, & x \in (\pi/2, \pi) \\ 0, & x \in (\pi, 3\pi/2) \\ \sin 2x, & x \in (3\pi/2, 2\pi) \end{cases}$$

the range of $f(x)$ is $(-1, 1)$.

Example 54 Find domain of $f(x)$, where

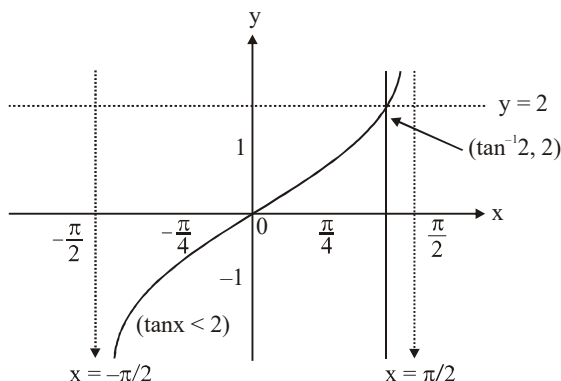
$$f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$$

Solution $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$

or $\sin x \geq 0$ and $16 - x^2 \geq 0$

or $2n\pi \leq x \leq (2n + 1)\pi$ and $-4 \leq x \leq 4$

Therefore, domain is $[-4, -\pi] \cup [0, \pi]$



7.3 POLYNOMIAL FUNCTION

If a function 'f' is called by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, \dots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n .

NOTE

1. A polynomial of degree one with no constant term is called an odd linear function. i.e., $f(x) = ax$, $a_0 \neq 0$
2. There are two polynomial functions, satisfying the relation $f(x) \cdot f(1/x) = f(x) + f(1/x)$. They are
 - (i) $f(x) = x^n + 1$
 - (ii) $f(x) = 1 - x^n$, where n is a positive integer

Proof : $f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$

$$f(x) \left(f\left(\frac{1}{x}\right) - 1 \right) = f\left(\frac{1}{x}\right) \quad \dots(i)$$

Replace x by $\frac{1}{x}$

$$f\left(\frac{1}{x}\right) \left(f(x) - 1 \right) = f(x) \quad \dots(ii)$$

Multiply (i) and (ii)

$$\left(f(x) - 1 \right) \left(f\left(\frac{1}{x}\right) - 1 \right) = 1$$

let $f(x) - 1 = g(x)$

$$g(x) \cdot g\left(\frac{1}{x}\right) = 1$$

So, $g(x) = \pm x^n$ (only possibility)

So $f(x) = 1 \pm x^n$

3. Domain of a polynomial function is \mathbb{R} .
4. Range of odd degree polynomial is \mathbb{R} whereas range of an even degree polynomial is never \mathbb{R} .

Example 55 If $f(x)$ is a polynomial satisfying $f(x) \cdot f(1/x) = f(x) + f(1/x)$ and $f(3) = 28$, then find the value of $f(4)$.

Solution (b)

$$f(x) = x^n + 1$$

or $f(3) = 3^n + 1 = 28$

or $3^n = 27$

$\therefore n = 3$

or $f(4) = 4^3 + 1 = 65$

7.4 ALGEBRAIC FUNCTION A function 'f' is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking radicals) starting with polynomials.

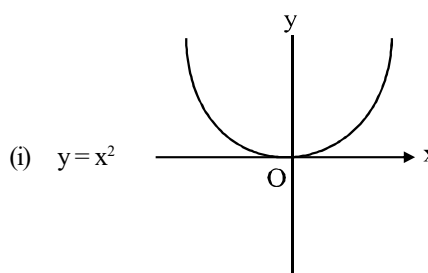
Examples : $f(x) = \sqrt{x^2 + 1}$;

$$g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

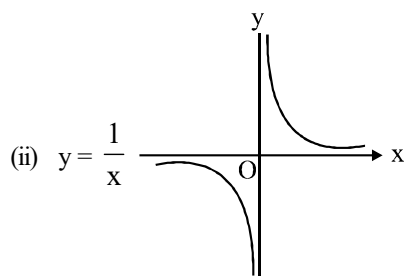
If y is an algebraic function of x , then it satisfies a polynomial equation of the form $P_0(x)y^n + P_1(x)y^{n-1} + \dots + P_{n-1}(x)y + P_n(x) = 0$, where 'n' is a positive integer and $P_0(x), P_1(x), \dots$ are polynomial in x .

Note that all polynomial functions are Algebraic but the converse is not true. A function that is not algebraic is called **TRANSCEDENTAL** function.

Basic algebraic function

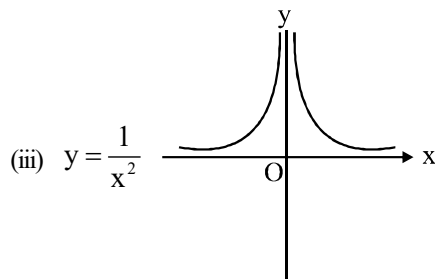


Domain : \mathbb{R} , **Range :** $\mathbb{R}^+ \cup \{0\}$ or $[0, \infty)$



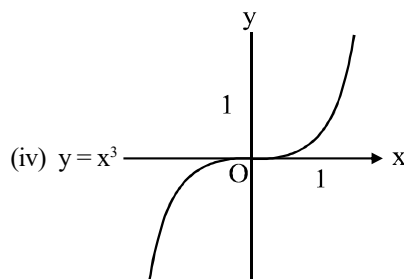
(ii) $y = \frac{1}{x}$

Domain : $\mathbb{R} - \{0\}$ or \mathbb{R}_0 , Range : $\mathbb{R} - \{0\}$



(iii) $y = \frac{1}{x^2}$

Domain : \mathbb{R}_0 , Range : \mathbb{R}^+ or $(0, \infty)$



(iv) $y = x^3$

Domain : \mathbb{R} , Range : \mathbb{R}

7.5 RATIONAL FUNCTION

A function that can be written as the quotient of two polynomial functions is said to be a rational function.

Let $P(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ and $Q(x) = b_0 + b_1x + b_2x^2 + \dots + b_mx^m$ be two polynomial functions.

Then the function f defined by $f(x) = \frac{P(x)}{Q(x)}$ is a rational function of x .

DPP 5

Total Marks 27

Time 20 Minute

- Question Number 1 to 4. **Marking Scheme** : +3 for correct answer 0 in all other cases. [(4×3=12)]
- Question Number 5. **Marking Scheme** : +3 for correct answer 0 in all other cases. [(5×3=15)]

- Solve $\sin x > -\frac{1}{2}$ or find the domain of

$$f(x) = \frac{1}{\sqrt{1 + 2 \sin x}}$$

2. Find the range of $f(x) = \frac{1}{2 \cos x - 1}$

3. Find the range of $f(x) = |\sin x| + |\cos x|$, $x \in \mathbb{R}$

4. Find domain of $f(x)$, where $f(x) = \frac{\sqrt{\cos x - \frac{1}{2}}}{\sqrt{6 + 35x - 6x^2}}$.

5. Find range of following

(i) $f(x) = \cos(2 \sin x)$

(ii) $f(x) = \cos^4 \frac{x}{5} - \sin^4 \frac{4x}{5}$

(iii) $f(x) = \sin \sqrt{x}$

(iv) $f(x) = \cot^2 \left(x - \frac{\pi}{4} \right)$

(v) $f(x) = \cos 2x - \sin 2x$

Result Analysis

- 20 to 27 Marks : **Advanced Level.**
- 12 to 19 Marks : **Mains Level.**
- < 11 Marks : **Please go through above article again.**

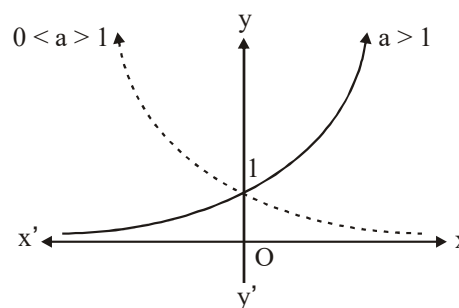
7.6 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

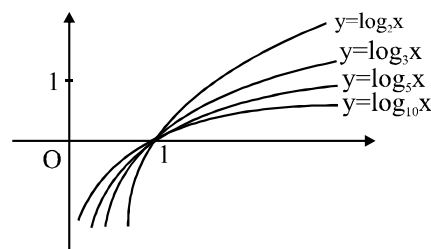
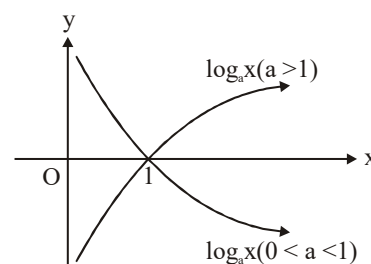
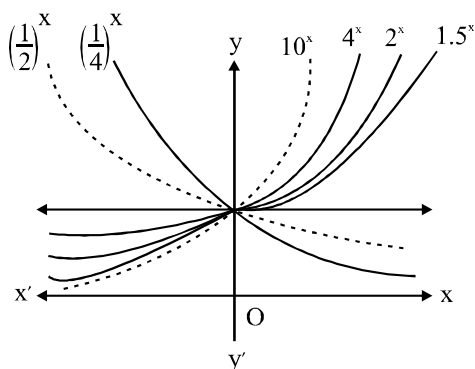
A. Exponential Function

$$y = a^x, a > 0, a \neq 1$$

Domain : \mathbb{R}

Range : $(0, \infty)$





Exponential Inequality

$$a^x > a^y = \begin{cases} x > y & \text{if } a > 1 \\ x < y & \text{if } 0 < a < 1 \end{cases} \quad \text{and}$$

$$a^x \geq a^y = \begin{cases} x \geq y & \text{if } a > 1 \\ x \leq y & \text{if } 0 < a < 1 \end{cases}$$

Example 56 Find the domain of

(i) $f(x) = \sqrt{3^x - 9}$ (ii) $f(x) = \sqrt{\left(\frac{1}{2}\right)^x - \frac{1}{8}}$

Solution (i) $f(x) = \sqrt{3^x - 9}$

We have, $3^x - 9 \geq 0 \Rightarrow 3^x \geq 9 \Rightarrow 3^x \geq 3^2$
 Here, $3 > 1$, So, $x \geq 2$

(ii) $f(x) = \sqrt{\left(\frac{1}{2}\right)^x - \frac{1}{8}}$

We have, $\left(\frac{1}{2}\right)^x - \frac{1}{8} \geq 0 \Rightarrow \left(\frac{1}{2}\right)^x \geq \left(\frac{1}{2}\right)^3$

Here, $0 < \frac{1}{2} < 1$ So, $x \leq 3$

B. Logarithmic Function

Logarithm function is the inverse of exponential function. Hence, the domain and range of the logarithmic functions are range and domain of exponential functions, respectively.

Also, the graph of the function can be obtained by taking the mirror image of the graph of the exponential function in the line $y = x$.

$y = \log_a x, a > 0$ and $a \neq 1$

Domain : $(0, \infty)$ Range : $(-\infty, \infty)$

Properties of Logarithmic Function

For $x, y > 0$ and $a > 0, a \neq 2$

1. $\log_a(x \cdot y) = \log_a x + \log_a y$
2. $\log_a(x/y) = \log_a x - \log_a y$
3. $\log_a(x^b) = b \log_a x$
4. $\log_{x^a} y^b = \frac{b}{a} \log_x y$
5. $\log_a x > \log_a y \Rightarrow \begin{cases} x > y, & \text{if } a > 1 \\ x < y, & \text{if } 0 < a < 1 \end{cases}$
6. $\log_a x = y \Rightarrow x = a^y$
7. If $\log_a x > y \Rightarrow \begin{cases} x > a^y, & \text{if } a > 1 \\ x < a^y, & \text{if } 0 < a < 1 \end{cases}$
8. $a^{\log_a x} = x$
9. $\log_y x = \frac{\log_a x}{\log_a y}$
10. $\log_a x > 0 \Rightarrow x > 1$ and $a > 1$ or $0 < x < 1$ and $0 < a < 1$

Example 57 Find the range of

- (a) $f(x) = \log_e \sin x$
- (b) $f(x) = \log_3(5 - 4x - x^2)$

Solution

(a) $f(x) = \log_e \sin x$ is defined if $\sin x \in (0, 1]$ for which $\log_e \sin x \in (-\infty, 0]$

(b) $f(x) = \log_3(5 - 4x - x^2) = \log_3\{9 - (x - 2)^2\}$
 $f(x)$ is defined if

$9 - (x - 2)^2 > 0$ but $9 - (x - 2)^2 \leq 9$
 or $0 < 9 - (x - 2)^2 \leq 9$
 or $-\infty < \log_3\{9 - (x - 2)^2\} \leq \log_3 9$

Hence, the range is $(-\infty, 2]$

Example 58 Find the domain of

$$f(x) = \sqrt{(0.625)^{4-3x} - (1.6)^{x(x+8)}}$$

Solution Clearly, $(0.625)^{4-3x} \geq (1.6)^{x(x+8)}$

$$\text{or } \left(\frac{5}{8}\right)^{4-3x} \geq \left(\frac{8}{5}\right)^{x(x+8)}$$

$$\text{or } \left(\frac{8}{5}\right)^{3x-4} \geq \left(\frac{8}{5}\right)^{x(x+8)}$$

$$\text{or } 3x-4 \geq x^2+8x$$

$$\text{or } x^2+5x+4 \leq 0$$

$$\text{or } -4 \leq x \leq -1$$

Hence, the domain of function $f(x)$ is $x \in [-4, -1]$

Example 59 Find the domain and range of

$$f(x) = \sqrt{\log_3 \{\cos(\sin x)\}}$$

Solution $f(x) = \sqrt{\log_3 \{\cos(\sin x)\}}$

$f(x)$ is defined only if $\log_3 \{\cos(\sin x)\} \geq 0$

$$\text{or } \cos(\sin x) \geq 1$$

$$\text{or } \cos(\sin x) = 1 \text{ as } -1 \leq \cos \theta \leq 1$$

$$\text{or } \sin x = 0 \text{ or } x = n\pi, n \in I$$

Hence, the domain consists of the multiples of π , i.e.,

$$\text{Domain : } \{n\pi, n \in I\}$$

Also, the range is $\{0\}$.

Example 60 Find the number of solutions of

$$(2013)^x + (2014)^x + (2015)^x - (2016)^x = 0$$

Solution $(2013)^x + (2014)^x + (2015)^x - (2016)^x = 0$

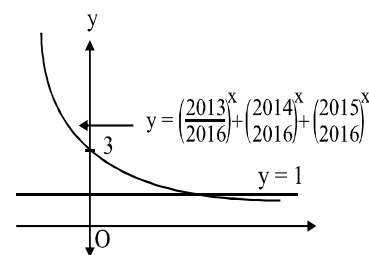
$$\text{or } (2013)^x + (2014)^x + (2015)^x = (2016)^x$$

$$\text{or } \left(\frac{2013}{2016}\right)^x + \left(\frac{2014}{2016}\right)^x + \left(\frac{2015}{2016}\right)^x = 1$$

Now, the number of solutions of the equation is equal to the number of times

$$y = \left(\frac{2013}{2016}\right)^x + \left(\frac{2014}{2016}\right)^x + \left(\frac{2015}{2016}\right)^x$$

and $y = 1$ intersect.



From the graph, the equation has only one solution.

Example 61 Find the domain of $f(x) = \sqrt{\log_{0.4} \left(\frac{x-1}{x+5}\right)}$

Solution $f(x) = \sqrt{\log_{0.4} \left(\frac{x-1}{x+5}\right)}$ exists if

$$\log_{0.4} \left(\frac{x-1}{x+5}\right) \geq 0 \text{ and } \left(\frac{x-1}{x+5}\right) > 0$$

$$\text{or } \frac{x-1}{x+5} \leq (0.4)^0 \text{ and } \frac{x-1}{x+5} > 0$$

$$\text{or } \frac{x-1}{x+5} < 1 \text{ and } \frac{x-1}{x+5} > 0$$

$$\text{or } \frac{x-1}{x+5} - 1 \leq 0 \text{ and } \frac{x-1}{x+5} > 0$$

$$\text{or } \frac{-6}{x+5} \leq 0 \text{ and } \frac{x-1}{x+5} > 0$$

$$\text{or } x+5 > 0 \text{ and } \frac{x-1}{x+5} > 0$$

$$\text{or } x > -5 \text{ and } x-1 > 0 \quad (\text{As } x+5 > 0)$$

$$\text{or } x > -5 \text{ and } x > 1$$

Thus, the domain $f(x)$ is $(1, \infty)$

Example 62 Find the range of $f(x) = \log_e x - \frac{(\log_e x)^2}{|\log_e x|}$

Solution $f(x) = \log_e x - \frac{(\log_e x)^2}{|\log_e x|}$

$$= \begin{cases} \log_e x - \frac{(\log_e x)^2}{(\log_e x)}, \log_e x > 0 \\ \log_e x - \frac{(\log_e x)^2}{(-\log_e x)}, \log_e x < 0 \end{cases}$$

$$= \begin{cases} 0, & x > 1 \\ 2 \log_e x, & 0 < x < 1 \end{cases}$$

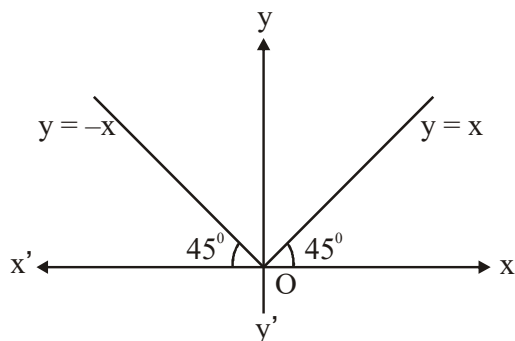
Therefore, range is $(-\infty, 0]$.

7.7 ABSOLUTE VALUE FUNCTION/MODULUS FUNCTIONS

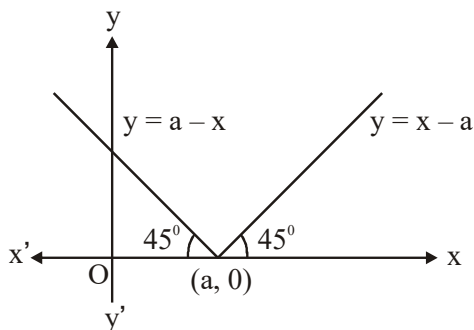
$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} = \sqrt{x^2} = \max \{x, -x\}$$

Domain : \mathbb{R}

Range : $[0, \infty)$



$$y = |x - a| = \begin{cases} x - a, & x \geq a \\ a - x, & x < a \end{cases}, \text{ where } a > 0$$



Properties Of Modulus Function

- (i) $|x| < a \Leftrightarrow -a < x < a$ i.e., $x \in (-a, a)$
 - (ii) $|x| \leq a \Leftrightarrow -a \leq x \leq a$ i.e., $x \in [-a, a]$
 - (iii) $|x| > a \Leftrightarrow x < -a$ or $x > a$ i.e., $x \in (-\infty, -a) \cup (a, \infty)$
 - (iv) $|x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$ i.e., $x \in (-\infty, -a] \cup [a, \infty)$
- If r is a positive real number and a is any real number, then
- (v) $|x - a| < r \Leftrightarrow a - r < x < a + r$ i.e., $x \in (a - r, a + r)$
 - (vi) $|x - a| \leq r \Leftrightarrow a - r \leq x \leq a + r$ i.e., $x \in [a - r, a + r]$
 - (vii) $|x - a| > r \Leftrightarrow x < a - r$ or $x > a + r$
i.e., $x \in (-\infty, a - r) \cup (a + r, \infty)$
 - (viii) $|x - a| \geq r \Leftrightarrow x \leq a - r$ or $x \geq a + r$
i.e., $x \in (-\infty, a - r] \cup [a + r, \infty)$
- If $a, b > 0$ and c are real numbers, then
- (ix) $a < |x| < b \Leftrightarrow x \in (-b, -a) \cup (a, b)$
 - (x) $a \leq |x| \leq b \Leftrightarrow x \in [-b, -a] \cup [a, b]$
 - (xi) $a \leq |x - c| \leq b \Leftrightarrow x \in [-b + c, -a + c] \cup [a + c, b + c]$
 - (xii) $a < |x - c| < b \Leftrightarrow x \in (-b + c, -a + c) \cup (a + c, b + c)$
 - (xiii) If $a, b, c \in \mathbb{R}$ such that $b^2 - 4ac < 0$, then
 - $a > 0 \Rightarrow ax^2 + bx + c > 0$ for all $x \in \mathbb{R}$
 - $a < 0 \Rightarrow ax^2 + bx + c < 0$ for all $x \in \mathbb{R}$

i.e., $ax^2 + bx + c$ and a are of the same sign for all $x \in \mathbb{R}$

Example 63 The solution set of the inequation $1 \leq |x - 2| \leq 3$ is

- (A) $[-1, 5]$
- (B) $[3, 5]$
- (C) $[-1, 1]$
- (D) $[-1, 1] \cup [3, 5]$

Solution (D)

We have
 $a \leq |x - c| \leq b \Leftrightarrow x \in [-b + c, -a + c] \cup [a + c, b + c]$
 $\therefore 1 \leq |x - 2| \leq 3 \Leftrightarrow x \in [-3 + 2, -1 + 2] \cup [1 + 2, 3 + 2]$
 $\Leftrightarrow x \in [-1, 1] \cup [3, 5]$

Example 64 The set of all values of x satisfying the inequations

- $|x - 1| \leq 5$ and $|x| \geq 2$ is.
- (A) $[-4, 6]$
 - (B) $[-4, -2]$
 - (C) $[-4, -2] \cup [2, 6]$
 - (D) $[2, 6]$

Solution (C)

We have,
 $|x - 1| \leq 5$ and $|x| \geq 2$
 $\Rightarrow -5 \leq x - 1 \leq 5$ and $(x \leq -2$ or $x \geq 2)$
 $\Rightarrow -4 \leq x \leq 6$ and $(x \leq -2$ or $x \geq 2)$
 $\Rightarrow x \in [-4, 6]$ and $x \in (-\infty, -2) \cup [2, \infty)$
 $\Rightarrow x \in [-4, -2] \cup [2, 6]$

Example 65 The solution set of the inequation $\frac{3}{|x| + 2} \geq 1$ is.

- (A) $[-1, 1]$
- (B) $(-1, 1)$
- (C) $(-\infty, 1]$
- (D) $[1, \infty)$

Solution (A)

We have
 $\frac{3}{|x| + 2} \geq 1$
 $\Rightarrow 3 \geq |x| + 2$
 [Multiplying both sides by $|x| + 2$ as it is positive]
 $\Rightarrow 1 \geq |x| \Rightarrow |x| \leq 1 \Rightarrow x \in [-1, 1]$

Example 66 Solve $|3x - 2| \leq \frac{1}{2}$.

Solution $|3x - 2| \leq \frac{1}{2}$

or $-\frac{1}{2} \leq 3x - 2 \leq \frac{1}{2}$

or $\frac{3}{2} \leq 3x \leq \frac{5}{2}$

or $\frac{1}{2} \leq x \leq \frac{5}{6}$

or $x \in \left[\frac{1}{2}, \frac{5}{6} \right]$

Example 67 If $\left| \frac{2}{x-4} \right| > 1$, then x belongs to the interval.

- (A) (2, 6) (B) (2, 4) \cup (4, 6)
 (C) $(-\infty, 2)$ (D) (6, ∞)

Solution (B)

We have,

$$\left| \frac{2}{x-4} \right| > 1$$

$$\Rightarrow \frac{2}{|x-4|} > 1$$

$$\Rightarrow 2 > |x-4|$$

$$\Rightarrow |x-4| < 2$$

$$\Rightarrow 4-2 < x < 4+2 \quad [\because |x-a| < r \Leftrightarrow a-r < x < a+r]$$

$$\Rightarrow 2 < x < 6$$

But, $\frac{2}{x-4}$ is not meaningful for $x = 4$.

$$\therefore 2 < x < 6 \text{ and } x \neq 4 \quad \Rightarrow x \in (2, 4) \cup (4, 6)$$

Example 68 Solve $\frac{-1}{|x|-2} \geq 1$, where $x \in \mathbb{R}$, $x \neq \pm 2$, to find

the domain of $f(x) = \sqrt{\frac{1-|x|}{|x|-2}}$.

Solution Given $f(x) = \sqrt{\frac{1-|x|}{|x|-2}}$

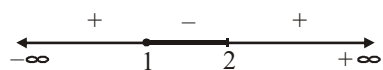
or $\frac{-1}{|x|-2} - 1 \geq 0$

or $\frac{-1-(|x|-2)}{|x|-2} \geq 0$

or $\frac{1-|x|}{|x|-2} \geq 0$

or $\frac{|x|-1}{|x|-2} \leq 0$

or $\frac{y-1}{y-2} \leq 0$, where $y = |x|$



or $1 \leq y < 2$

or $1 \leq x < 2$

or $x \in (-2, -1] \cup [1, 2)$

Example 69 Solve $|x-1| + |x-2| \geq 4$.

Solution Let $f(x) = |x-1| + |x-2|$

First mark all critical points i.e., where all Mod. become zero.

Here such points are $x = 1, 2$

mark these two points on number line

Now, identify all three interval and defined $f(x)$ accordingly

$$f(x) = \begin{cases} 1-x+2-x, & x < 1 \\ x-1+2-x, & 1 \leq x \leq 2 \\ x-1+x-2, & x > 2 \end{cases}$$

$$f(x) = \begin{cases} 3-2x, & x < 1 \\ 1, & 1 \leq x \leq 2 \\ 2x-3, & x > 2 \end{cases}$$

Now,

$$f(x) \geq 4 \Rightarrow \begin{cases} 3-2x \geq 4 & \text{and } x < 1 \\ 1 \geq 4 & \text{and } 1 \leq x \leq 2 \\ 2x-3 \geq 4 & \text{and } x > 2 \end{cases}$$

$$\Rightarrow \begin{cases} x \leq -\frac{1}{2} & \text{and } x < 1 \\ \text{NOT POSSIBLE} \\ x \geq \frac{7}{2} & \text{and } x > 2 \end{cases}$$

Hence, the solution is $x \in \left(-\infty, -\frac{1}{2} \right] \cup \left[\frac{7}{2}, \infty \right)$

Example 70 Solve $|\sin x + \cos x| = |\sin x| + |\cos x|$, $x \in [0, 2\pi]$.

Note : $|A| + |B| = |A+B|$ is true only when $A \cdot B \geq 0$

Solution The given relation holds only when $\sin x$ and $\cos x$ have the same sign or at least one of them is zero.

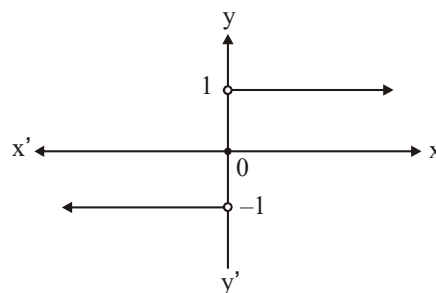
Hence, $x \in [0, \pi/2] \cup [\pi, 3\pi/2] \cup \{2\pi\}$

Example 71 Solve $|\cot x + \operatorname{cosec} x| = |\cot x| + |\operatorname{cosec} x|$.

Solution We know $|A| + |B| = |A+B|$ is true only when $A \cdot B \geq 0$

Hence, $x \in \left(0, \frac{\pi}{2} \right] \cup \left[\frac{3\pi}{2}, 2\pi \right)$

7.8 SIGNUM FUNCTION



$$y = f(x) = \text{sgn}(x)$$

$$\text{sgn}(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{or} \quad \text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Domain : R

Range : $\{-1, 0, 1\}$

$$\text{In general, } \text{sgn}(f(x)) = \begin{cases} \frac{|f(x)|}{f(x)}, & f(x) \neq 0 \\ 0, & f(x) = 0 \end{cases}$$

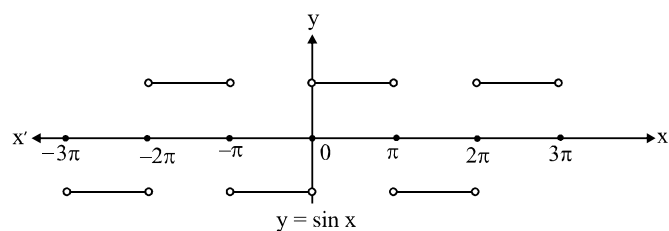
$$\text{or } \text{sgn}(f(x)) = \begin{cases} -1, & f(x) < 0 \\ 0, & f(x) = 0 \\ 1, & f(x) > 0 \end{cases}$$

Example 72 Write the equivalent (piecewise) definition of $f(x) = \text{sgn}(\sin x)$.

$$\text{Solution } \text{sgn}(\sin x) = \begin{cases} -1, & \sin x < 0 \\ 0, & \sin x = 0 \\ 1, & \sin x > 0 \end{cases}$$

$$= \begin{cases} \sin x < 0 & x \in \{(-3\pi, -2\pi), (-\pi, 0), (\pi, 2\pi), (3\pi, 4\pi), \dots\} \\ \sin x > 0 & x \in \{(-2\pi, -\pi), (0, \pi), (2\pi, 3\pi), \dots\} \\ \sin x = 0 & x \in \{(-2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \dots)\} \end{cases}$$

$$= \begin{cases} -1, & x \in ((2n+1)\pi, (2n+2)\pi), n \in \mathbb{Z} \\ 0, & x = n\pi, n \in \mathbb{Z} \\ 1, & x \in (2n\pi, (2n+1)\pi), n \in \mathbb{Z} \end{cases}$$



Example 73 Find the range of $f(x) = \text{sgn}(x^2 - 2x + 3)$.

$$\text{Solution } x^2 - 2x + 3 = (x-1)^2 + 1 > 0 \quad \forall x \in \mathbb{R}$$

$$\text{or } f(x) = \text{sgn}(x^2 - 2x + 3)$$

Hence, the range is $\{1\}$.

Example 74 Write the equivalent definition of the following functions.

$$(a) f(x) = \text{sgn}(\log_e |x|) \quad (b) f(x) = \text{sgn}(x^3 - x)$$

Solution

$$(a) f(x) = \text{sgn}(\log_e |x|)$$

$\log_e |x|$ can be positive, negative or zero. So we will find corresponding values of x for which has this can be

positive, negative or zero.

$$(i) \text{ first solve, } \log_e |x| > 0$$

$$\begin{aligned} & |x| > 1 \\ \text{Sgn (positive)} &= 1, |x| > 1 \end{aligned} \quad \dots(1)$$

$$(ii) \log_e |x| < 0$$

$$\begin{aligned} & |x| < 1 \\ \text{Sgn (negative)} &= -1, |x| < 1 \end{aligned} \quad \dots(2)$$

$$(iii) \log_e |x| = 0$$

$$\begin{aligned} & |x| = 1 \\ \text{Sgn (zero)} &= 0, |x| = 1 \end{aligned} \quad \dots(3)$$

Combine (1), (2) and (3)

$$\text{Sgn}(\log_e |x|) = \begin{cases} 1, & |x| > 1 \\ -1, & |x| < 1 \\ 0, & |x| = 1 \end{cases}$$

$$(b) \text{ Sgn}(x^3 - x)$$

$$\text{consider } f(x) = x^3 - x = x(x^2 - 1) = x(x+1)(x-1)$$

$$f(x) > 0, x \in (-1, 0) \cup (1, \infty)$$

$$f(x) < 0, x \in (-\infty, -1) \cup (0, 1)$$

$$f(x) = 0, x = -1, 0, 1$$

$$\text{Sgn}(x^3 - x) = \begin{cases} 1 & x \in (-1, 0) \cup (1, \infty) \\ -1 & x \in (-\infty, -1) \cup (0, 1) \\ 0 & x = -1, 0, 1 \end{cases}$$

Example 75 Find the possible value of following

$$(i) \text{ Sgn}(\ln(x^2 - x + 2)) \quad (ii) \text{ Sgn}(\ln(x^2 - x + 1))$$

Solution

$$(i) y = f(x) = \text{Sgn}(\ln(x^2 - x + 2))$$

domain of given function is R

For Range :

$$y = \text{Sgn}(\ln(x^2 - x + 2))$$

$$y = \text{Sgn}\left(\ln\left(\left(x - \frac{1}{2}\right)^2 + \frac{7}{4}\right)\right)$$

$$\Rightarrow \text{since } \left(x - \frac{1}{2}\right)^2 \geq 0, x \in \mathbb{R}$$

$$\Rightarrow \text{Sgn}\left(\ln\left(\left[\frac{7}{4}, \infty\right)\right)\right)$$

$$\Rightarrow \ln x > 0, \text{ if } x > 1$$

$$\Rightarrow \text{Here } \frac{7}{4} > 1, \text{ So } \left[\frac{7}{4}, \infty\right) \text{ is always positive}$$

$$\Rightarrow \text{Sgn (positive)} = 1$$

$$\text{Range} = \{1\}$$

$$(ii) y = \text{Sgn}(\ln(x^2 - x + 1))$$

$$y = \text{Sgn} \left(\ln \left(\left(x - \frac{1}{2} \right)^2 + \frac{3}{4} \right) \right)$$

$$\Rightarrow \left(x - \frac{1}{2} \right)^2 \geq 0, x \in \mathbb{R}$$

$$\Rightarrow \text{Sgn} \left(\ln \left[\frac{3}{4}, \infty \right) \right)$$

$$\Rightarrow \ln \left[\frac{3}{4}, \infty \right) \text{ can be positive, negative or zero also}$$

$$\Rightarrow \text{Sgn}(\text{Positive, negative or zero}) = -1, 0, 1$$

$$\text{Range} = \{1, -1, 0\}$$

Example 76 Find domain of following

(i) $f(x) = \sqrt{\log_x(\cos 2\pi x)}$

(ii) $f(x) = \sqrt{(x^2 - 3x - 10)\ln^2(x - 3)}$

Solution

(i) $f(x) = \sqrt{\log_x(\cos 2\pi x)}$

Case I

$$0 < x < 1$$

$$\log_x \cos(2\pi x) \geq 0 \text{ and } \cos(2\pi x) > 0$$

$$\cos(2\pi x) \leq 1 \text{ and } \cos(2\pi x) > 0$$

$$0 < \cos(2\pi x) \leq 1$$

$$\left(0 \leq 2\pi x < \frac{\pi}{2} \right) \cup \left(\frac{3\pi}{2} < 2\pi x \leq 2\pi \right)$$

$$\left(2n\pi \leq 2\pi x < (2n+1)\frac{\pi}{2} \right) \cup \left(2n\pi + \frac{3\pi}{2} < 2\pi x \leq 2n\pi + 2\pi \right),$$

$$n \in \mathbb{I}$$

$$\left(n \leq x < n + \frac{1}{4} \right) \cup \left(n + \frac{3}{4} < x \leq n + 1 \right)$$

for $n = 0$

$$\left(0 \leq x < \frac{1}{4} \right) \cup \left(\frac{3}{4} < x \leq 1 \right) \dots(i)$$

Case II

$$x > 1$$

$$\log_x \cos(2\pi x) \geq 0$$

$$\cos(2\pi x) \geq 1$$

$$\cos 2\pi x = 1$$

$$x \in \mathbb{N} \text{ and } x \geq 2 \quad (\because x > 1) \dots(ii)$$

from (i) and (ii)

$$x \in \left[0, \frac{1}{4} \right) \cup \left(\frac{3}{4}, 1 \right] \cup x \in \mathbb{N}, x \geq 2$$

(ii) $\sqrt{(x^2 - 3x - 10)\ln^2(x - 3)}$

$$(x - 3) > 0, \quad x^2 - 3x - 10 \geq 0$$

$$x > 3 \quad (x + 2)(x - 5) \geq 0$$

$$(-\infty, -2] \cup [5, \infty)$$

Include $x = 4$ also

\because at $x = 4$, the value of $\ln^2(x - 3) = 0$

although $(x^2 - 3x - 10) = -ve$

$$\text{domain} [5, \infty) \cup \{4\}$$

Example 77 Find the domain & range of the following functions.

(i) $y = \log_{\sqrt{5}}(\sqrt{2}(\sin x - \cos x) + 3)$

(ii) $f(x) = \frac{x}{1 + |x|}$ (iii) $f(x) = \frac{x^2 - 3x + 2}{x^2 + x - 6}$

(iv) $f(x) = \sqrt{x^2 - |x|} + \frac{1}{\sqrt{9 - x^2}}$

Solution

(i) $y = \log_{\sqrt{5}}(\sqrt{2}(\sin x - \cos x) + 3)$

Domain of function is $x \in \mathbb{R}$

for Range

$$y = \log_{\sqrt{5}}(\sqrt{2}(\sin x - \cos x) + 3)$$

$$= \log_{\sqrt{5}}(\sqrt{2}(-\sqrt{2} \text{ to } \sqrt{2}) + 3)$$

$$= \log_{\sqrt{5}}(-2 \text{ to } 2) + 3$$

$$= \log_{\sqrt{5}}(1 \text{ to } 5)$$

$$y = 0 \text{ to } 2$$

(ii) $f(x) = \frac{x}{1 + |x|}$

Domain : $1 + |x| \neq 0$

$$|x| \neq -1 \text{ is true for } x \in \mathbb{R}$$

So, domain of $f(x)$ is \mathbb{R} .

Range :

Case : 1 $x \geq 0$

Case : 2 $x \leq 0$

$$y = \frac{x}{1 + x}$$

$$y = \frac{x}{1 - x}$$

$$y = 1 - \frac{1}{1 + x}$$

$$y = \frac{1}{1 - x} - 1$$

$$y = 1 - \frac{1}{1 + [0 \text{ to } \infty)}$$

$$y = \frac{1}{1 - (-\infty \text{ to } 0]} - 1$$

$$y = 1 - \frac{1}{[1 \text{ to } \infty)}$$

$$y = \frac{1}{(\infty \text{ to } 1]} - 1$$

$$y = 1 - [1 \text{ to } 0)$$

$$y \in [0, 1)$$

$$y = (0 \text{ to } 1] - 1$$

$$y = (-1 \text{ to } 0]$$

So Range of $f(x)$ is $(-1, 1)$

$$y \in (-1, 0]$$

(iii) $f(x) = \frac{x^2 - 3x + 2}{x^2 + x - 6}$

$$D_f \in \mathbb{R} - \{3, -2\}$$

for range $y = \frac{(x-1)(x-2)}{(x+3)(x-2)}$

$$y = \frac{x-1}{x+3}$$

....(i)

$$x = \frac{3y+1}{1-y}$$

$$y \neq 1$$

put $x = 2$, in equation (i)

$$y \neq \frac{1}{5}$$

So the range $y \in \mathbb{R} - \left\{1, \frac{1}{5}\right\}$

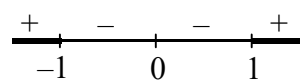
(iv) $f(x) = \sqrt{x^2 - |x|} + \frac{1}{\sqrt{9 - x^2}}$

$$x^2 - |x| > 0 \quad 9 - x^2 > 0$$

$$x^2 > |x| \quad -3 < x < 3$$

$$x^4 > x^2$$

$$x^2(x^2 - 1) \geq 0$$



$$x \in (-3, -1] \cup [1, 3) \cup \{0\}$$

1. Find the domain of $f(x) = \sqrt{\left(\frac{1-5^x}{7^{-x}-7}\right)}$.

2. Find the domain of function $f(x) = \log_4[\log_3\{\log_3(18-x^2-77)\}]$

3. Let $x \in \left(0, \frac{\pi}{2}\right)$. Then find the domain of the function

$$f(x) = \frac{1}{\sqrt{-\log_{\sin x} \tan x}}$$

4. Solve $\|x - 1| - 5| \geq 2$

5. Solve $\frac{|x+3|+x}{x+2} > 1$.

6. Solve $|-2x^2 + 1 + e^x + \sin x| = |2x^2 - 1| + e^x + |\sin x|$, $x \in [0, 2\pi]$

7. Verify that

(i) $x \operatorname{sgn} x = |x|$

(ii) $|x| \operatorname{sgn} x = x$

(iii) $x (\operatorname{sgn} x) (\operatorname{sgn} x) = x$

8. Find domain of $f(x) = \sqrt{\log_2 \left(\frac{5x - x^2}{4}\right)}$

9. Find the domain of $f(x) = \sqrt{\frac{\sqrt{x+1}}{x-1}}$

10. Find Range of following

(i) $\ln(5x^2 - 8x + 4)$

(ii) $\log_{\sqrt{2}}(2 - \log_2(16 \sin^2 x + 1))$

Result Analysis

1. 28 to 36 Marks : **Advanced Level.**

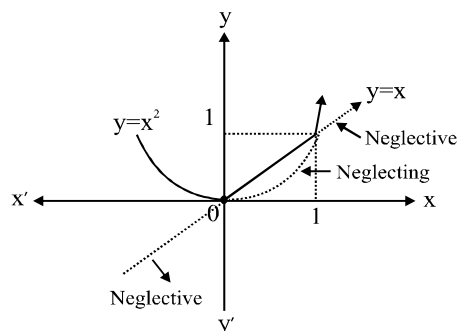
2. 20 to 28 Marks : **Mains Level.**

3. < 20 Marks : **Please go through above article again.**

7.9 Function Of The Form $f(x) = \max. \{g_1(x), g_2(x), \dots, g_n(x)\}$ or $f(x) = \min. \{g_1(x), g_2(x), \dots, g_n(x)\}$

Let us consider the function $f(x) = \max. \{x, x^2\}$.

To write the equivalent definition of the function, first draw the graph of $y = x$ and $y = x^2$.



Now, from the graph, we can see that

1. For $x \in (-\infty, 0)$, the graph of $y = x^2$ lies above the graph

DPP 6

Total Marks 36

Time 30 Minute

- Question Number 1 to 6 . **Marking Scheme** : +3 for correct answer -1 in all other cases. [(6.3)=18]
- Question Number 7 . **Marking Scheme** : +3 for correct answer 0 in all other cases. [(3.2)=6]
- Question Number 8 . **Marking Scheme** : +3 for correct answer 0 in all other cases. [(1.3)=3]
- Question Number 9 . **Marking Scheme** : +3 for correct answer 0 in all other cases. [(1.3)=3]
- Question Number 10. **Marking Scheme** : +3 for correct answer 0 in all other cases. [(2.3)=6]

of $y = x$ or $x^2 > x$.

2. For $x \in (0, 1)$, the graph of $y = x$ lies above the graph of $y = x^2$ or $x > x^2$.
3. For $x \in (1, \infty)$, the graph of $y = x^2$ lies above the graph of $y = x$ or $x^2 > x$.

Hence, we have

$$f(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \leq x \leq 1 \\ x^2, & x > 1 \end{cases}$$

For $f(x) = \min. \{x, x^2\}$, we have

$$f(x) = \begin{cases} x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ x, & x > 1 \end{cases}$$

Example 78 Consider the function

$$f(x) = \max. \{1, |x - 1|\}, \quad \min = \{4, |3x - 1|\} \quad \forall x \in \mathbb{R}$$

Then, find the value of $f(3)$.

Solution $f(3) = \max. \{1, |3 - 1|\}, \min. \{4, |9 - 1|\}$
 $= \max. \{1, 2, 4\}$
 $= 4$

Example 79 If $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ are two given functions, then prove that

$$2 \min. \{f(x) - g(x), 0\} = f(x) - g(x) - |g(x) - f(x)|$$

Solution $h(x) = 2 \min. \{f(x) - g(x), 0\}$

$$= \begin{cases} 0, & f(x) > g(x) \\ 2\{f(x) - g(x)\} & f(x) \leq g(x) \end{cases}$$

$$= \begin{cases} f(x) - g(x) - |f(x) - g(x)|, & f(x) > g(x) \\ f(x) - g(x) - |f(x) - g(x)|, & f(x) \leq g(x) \end{cases}$$

$$\therefore h(x) = f(x) - g(x) - |g(x) - f(x)|$$

Example 80 Draw the graph of the function $f(x) = \max. \{\sin x,$

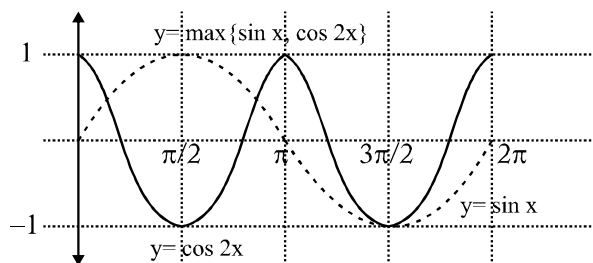
$\cos 2x\}$, $x \in [0, 2\pi]$. Write the equivalent

definition of $f(x)$ and find the range of the function.

Solution $\sin x = \cos 2x$
 or $\sin x = 1 - 2 \sin^2 x$
 or $2 \sin^2 x + \sin x - 1 = 0$
 or $(2 \sin x - 1)(\sin x + 1) = 0$

i.e., $\sin x = \frac{1}{2}$ or $\sin x = -1$

i.e., $x = \frac{\pi}{6}, \frac{5\pi}{6}$ or $x = \pi$



From the graph,

$$f(x) = \begin{cases} \cos 2x, & 0 \leq x < \frac{\pi}{6} \\ \sin x, & \frac{\pi}{6} \leq x < \frac{5\pi}{6} \\ \cos 2x, & \frac{5\pi}{6} < x \leq 2\pi \end{cases}$$

Also, the range of the function is $[-1, 1]$

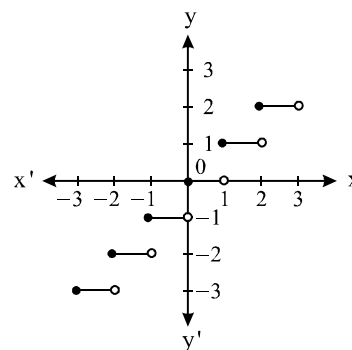
7.10 GREATEST INTEGER AND FRACTIONAL PART FUNCTION

A. Greatest Integer Function :

For any real number x , we denote $[x]$, the greatest integer less than or equal to x .

For example, $[2.45] = 2, [-2.1] = -3, [1.75] = 1, [0.32] = 0$, etc. The function f defined by $f(x) = [x]$ for all $x \in \mathbb{R}$, is called the greatest integer function.

Clearly, the domain of the greatest integer function is the set \mathbb{R} of all real numbers and the range is the set of all integers as it attains only integral values. The graph of the greatest integer function is showing in fig.



In general, if n is an integer and x is any number satisfying $n \leq x < n + 1$, then

$$[x] = n$$

Also, if $\{x\}$ denotes the fractional part of x , then $[x] = x - \{x\}$ or $x = [x] + \{x\}$.

Properties of greatest integer function :

If n is an integer and x is any real number between n and $n + 1$, then the greatest integer function has the following properties.

- (i) $[-n] = -[n]$

(ii) $[x + n] = [x] + n$

(iii) $[-x] = -[x] - 1$

(iv) $[x] + [-x] = \begin{cases} -1, & \text{if } x \notin \mathbb{Z} \\ 0, & \text{if } x \in \mathbb{Z} \end{cases}$

(v) $[x] - [-x] = \begin{cases} 2[x], & \text{if } x \in \mathbb{Z} \\ 2[x] + 1, & \text{if } x \notin \mathbb{Z} \end{cases}$

(vi) $[x] \geq n \Rightarrow x \geq n, \text{ where } n \in \mathbb{Z}$

(vii) $[x] \leq n \Rightarrow x < n + 1, n \in \mathbb{Z}$

(viii) $[x] > n \Rightarrow x \geq n + 1, n \in \mathbb{Z}$

(ix) $[x] < n \Rightarrow x < n, n \in \mathbb{Z}$

(x) $[x + y] = [x] + [y + x - [x]]$ for all $x, y \in \mathbb{R}$

(xi) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \dots + \left[x + \frac{n-1}{n}\right] = [nx], n \in \mathbb{N}$

NOTE:

$$f(x) = \frac{1}{[x]}$$

Domain : $\mathbb{R} - [0, 1)$

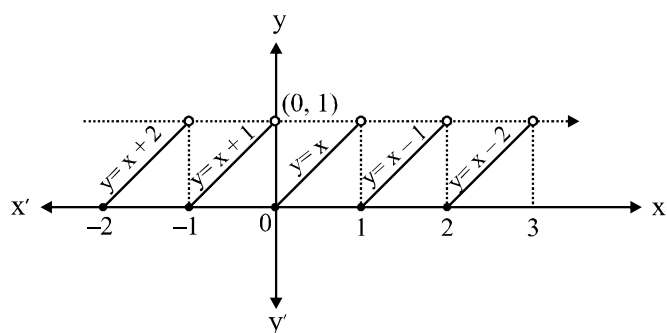
Range : $\{x \mid x = \frac{1}{n}, n \in \mathbb{I}_0\}$

B. Fractional Part Function

$$y = f(x) = \{x\} = x - [x]$$

$$y = \{x\} = x - [x] = \begin{cases} x - (-1) & , -1 \leq x < 0 \\ x - 0 & , 0 \leq x < 1 \\ x - 1 & , 1 \leq x < 2 \end{cases}$$

$$y = \{x\} = \begin{cases} x + 1 & , -1 \leq x < 0 \\ x & , 0 \leq x < 1 \\ x - 1 & , 1 \leq x < 2 \end{cases}$$



Properties of Fractional Part Function :

1. Domain : \mathbb{R}
2. Range : $[0, 1)$
3. $[x + y] = [x] + [y], 0 \leq \{x\} + \{y\} < 1$
4. $[x + y] = [x] + [y] + 1, 1 \leq \{x\} + \{y\} < 2$
5. $\{x\} + \{-x\} = 0$ if $x \in \mathbb{I}$
6. $\{x\} + \{-x\} = 1$ if $x \notin \mathbb{I}$

Example 81 Find the domain of

$$f(x) = \sqrt{([x] - 1)} + \sqrt{4 - [x]}$$

(Where $[]$ represents the greatest integer function).

Solution Given $f(x) = \sqrt{([x] - 1)} + \sqrt{4 - [x]}$

So, $f(x)$ is defined when $[x] - 1 \geq 0$ and $4 - [x] \geq 0$. i.e.,
 $1 \leq [x] \leq 4$ or $1 \leq x < 5$

Hence, the domain of $f(x)$ is $D_f = [1, 5)$

Example 82 Solve $2[x] = x + \{x\}$, where $[.]$ and $\{.\}$ denote the greatest integer function and the fractional part function, respectively.

Solution Given $2[x] = x + \{x\}$

or $2[x] = [x] + 2\{x\}$

or $\{x\} = \frac{[x]}{2}$

or $0 \leq \frac{[x]}{2} < 1$

or $0 \leq [x] < 2$

or $[x] = 0, 1$

For $[x] = 0$, we get $\{x\} = 0$ or $x = 0$

For $[x] = 1$, we get $\{x\} = \frac{1}{2}$ or $x = \frac{3}{2}$

Example 83 Solve the system of equations in $x, y,$ and z satisfying the following equations :

$$x + [y] + \{z\} = 3.1$$

$$\{x\} + y + [z] = 4.3$$

$$[x] + \{y\} + [z] = 5.4$$

where $[.]$ denotes the greatest integer function and $\{.\}$ denotes the fractional part function).

Solution Adding all the three equations, we get

$$2(x + y + z) = 12.8 \text{ or } x + y + z = 6.4 \quad \dots(i)$$

Adding the first two equations, we get

$$x + y + z + [y] + \{x\} = 7.4 \quad \dots(ii)$$

Adding the second and third equations, we get

$$x + y + z + [z] + \{y\} = 9.7 \quad \dots(iii)$$

Adding the first and fourth equations, we get

$$x + y + z + [x] + \{z\} = 8.5 \quad \dots(iv)$$

From (i) and (ii), $[y] + \{x\} = 1$

From (i) and (iii), $[z] + \{y\} = 3.3$

From (i) and (iv), $[x] + \{z\} = 2.1$

So,

$$[x] = 2, [y] = 1, [z] = 3, \{x\} = 0, \{y\} = 0.3 \text{ and } \{z\} = 0.1$$

$$\therefore x = 2, y = 1.3, z = 3.1$$

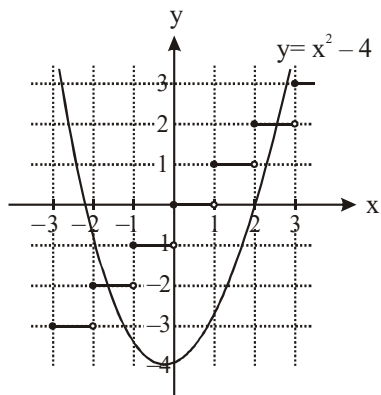
Example 84 Solve $x^2 - 4 - [x] = 0$ (where $[.]$ denotes the greatest

integer function).

Solution The best method to solve such system is graphical one.

The given equation is $x^2 - 4 = [x]$.

Then, the solutions of the equation are the values of x where $y = x^2 - 4$ and $y = [x]$ intersect.



From the graph, it is seen that these intersect when

$$x^2 - 4 = 2 \text{ and } x^2 - 4 = -2$$

i.e., $x^2 = 6$ or $x^2 = 2$

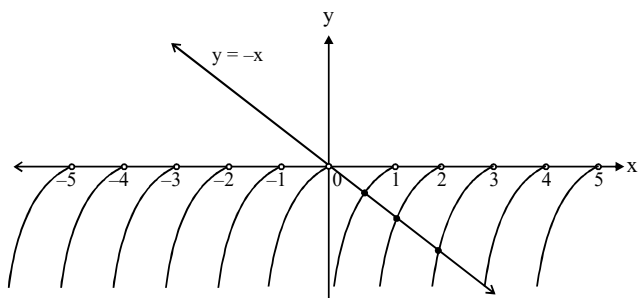
i.e., $x = \sqrt{6}$ or $-\sqrt{2}$

Example 85 $\log_{10} \{x\} = -x$, find no of solution $x \in (-3, 3)$.

Solution Two different function one linear and other logarithmic are given. So we will solve such kind of problem using graph.

So we will draw graph of $y = -x$ and $y = \log_{10} \{x\}$.

While drawing graph of $y = \log_{10} \{x\}$, we will take care as $0 \leq \{x\} < 1, \forall x \in \mathbb{R}$



So the no. of solution is 3

Example 86 If $f(x) = \begin{cases} [x], & 0 \leq \{x\} < 0.5 \\ [x] + 1, & 0.5 < \{x\} < 1 \end{cases}$ then prove that

$f(x) = -f(-x)$ (where, $[.]$ and $\{.\}$ represent the greatest integer function and the fractional part function, respectively).

Solution $f(-x) = \begin{cases} [-x], & 0 \leq \{-x\} < 0.5 \\ [-x] + 1, & 0.5 < \{-x\} < 1 \end{cases}$

$$= \begin{cases} [-x], & \{-x\} = 0 \\ [-x], & 0 < \{-x\} < 0.5 \\ [-x] + 1, & 0.5 < \{-x\} < 1 \end{cases}$$

$$= \begin{cases} -[x], & \{x\} = 0 \\ -1 - [x], & 0 < 1 - \{x\} < 0.5 \\ -1 - [x] + 1, & 0.5 < 1 - \{x\} < 1 \end{cases}$$

$$= \begin{cases} -[x], & \{x\} = 0 \\ -1 - [x], & 0.5 < \{x\} < 1 \\ -[x], & 0 < \{x\} < 0.5 \end{cases}$$

$$= \begin{cases} -[x], & 0 \leq \{x\} < 0.5 \\ -1 - [x], & 0.5 < \{x\} < 1 \end{cases}$$

$$= \begin{cases} -[x], & 0 \leq \{x\} < 0.5 \\ 1 + [x], & 0.5 < \{x\} < 1 \end{cases} = -f(x)$$

Example 87 Let $[k]$ denotes the greatest integer less than or equal to k .

If number of positive integral solutions of the equation

$$\left[\frac{x}{[\pi^2]} \right] = \left[\frac{x}{\left[\frac{11}{2} \right]} \right]$$

is n , then find the value of $\sqrt{n-8}$.

Solution $\left[\frac{x}{9} \right] = \left[\frac{x}{11} \right]$

Case I $0 \leq \frac{x}{9} < 1$ and $0 \leq \frac{x}{11} < 1$

$\Rightarrow 0 \leq x < 9$ and $0 \leq x < 11$

\Rightarrow common value of x is $\{1, 2, 3, \dots, 8\}$

Case II $1 \leq \frac{x}{9} < 2$ and $1 \leq \frac{x}{11} < 2$

$\Rightarrow 9 \leq x < 18$ and $11 \leq x < 22$

$\Rightarrow x \in \{11, 12, \dots, 17\}$

Case III $2 \leq \frac{x}{9} < 3$ and $2 \leq \frac{x}{11} < 3$

$\Rightarrow 18 \leq x < 27$ and $22 \leq x < 33$

$\Rightarrow x \in \{22, 23, \dots, 26\}$

Case IV $3 \leq \frac{x}{9} < 4$ and $3 \leq \frac{x}{11} < 4$

$\Rightarrow 27 \leq x < 36$ and $33 \leq x < 44$

$\Rightarrow x \in \{33, 34, \dots, 35\}$

Case V $4 \leq \frac{x}{9} < 5$ and $4 \leq \frac{x}{11} < 5$

$\Rightarrow x = 44$

∴ total positive integer
 $x = 8 + 7 + 5 + 3 + 1 = 24$
 ∴ Answer = $\sqrt{24 - 8} = 4$

Example 88 Find the domain and range of

$$f(x) = \log_{(\operatorname{cosec} x - 1)} (2 - [\sin x] - [\sin x]^2)$$

Solution We have, $f(x) = \log_{(\operatorname{cosec} x - 1)} (2 - [\sin x] - [\sin x]^2)$

for Domain

$$\begin{aligned} \operatorname{cosec} x - 1 &> 0 \\ \operatorname{cosec} x &> 1 && \dots \text{(i)} \\ \operatorname{cosec} x - 1 &\neq 1 \\ \operatorname{cosec} x &\neq 2 \end{aligned}$$

$$x \neq \frac{\pi}{6}, \frac{5\pi}{6} \quad \dots \text{(ii)}$$

$$\sin x \neq 1$$

$$x \neq \frac{\pi}{2} \quad \dots \text{(iii)}$$

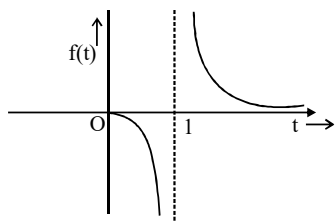
from (i), (ii) & (iii)

$$x \in (2n\pi, 2n\pi + \pi) - \left\{ 2n\pi + \frac{\pi}{6}, 2n\pi + \frac{\pi}{2}, 2n\pi + \frac{5\pi}{6} \right\}$$

for Range

$$\log_t 2 = f(x), \quad t = \operatorname{cosec} x - 1$$

$$f(t) = \frac{1}{\log_2 t} \quad t \in (0, \infty) - \{1\}$$



Range of function is $(-\infty, \infty) - \{0\}$.

7.11 IDENTICAL FUNCTION

Two functions f and g are said to be identical if

1. Domain of f = Domain of g , i.e., $D_f = D_g$.
2. The range of f = Range of g .
3. $f(x) = g(x) \quad \forall x \in D_f \text{ or } x \in D_g$

For example, $f(x) = x$ and $g(x) = \sqrt{x^2}$ are not identical functions as $D_f = D_g$ but $R_f = \mathbb{R}$, $R_g = [0, \infty)$.

Example 89 Find the values of x for which the following functions are identical.

(a) $f(x) = x$ and $g(x) = \frac{1}{1/x}$

(b) $f(x) = \cos x$ and $g(x) = \frac{1}{\sqrt{1 + \tan^2 x}}$

(c) $f(x) = \frac{\sqrt{9 - x^2}}{\sqrt{x - 2}}$ and $g(x) = \sqrt{\frac{9 - x^2}{x - 2}}$

Solution

(a) $f(x) = x$ is defined for all x . But

$$g(x) = \frac{1}{1/x} = x$$

is not defined for $x = 0$ as $1/x$ is not defined at $x = 0$. Hence, both the functions are identical for $x \in \mathbb{R} - \{0\}$.

(b) $f(x) = \cos x$ has domain \mathbb{R} and range $[-1, 1]$.

$$\text{But } g(x) = \frac{1}{\sqrt{1 + \tan^2 x}} = \frac{1}{\sqrt{\sec^2 x}} = |\cos x|$$

has domain $\mathbb{R} - \{(2n + 1)\pi/2, n \in \mathbb{Z}\}$ as $\tan x$ is not defined for $x = (2n + 1)\pi/2, n \in \mathbb{Z}$.

Also, the range of $g(x) = |\cos x|$ is $[0, 1]$.

Hence, $f(x)$ and $g(x)$ are identical if x lies in the first and fourth quadrants, i.e.,

$$x \in \left(-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi \right), n \in \mathbb{Z}$$

(c) $f(x) = \frac{\sqrt{9 - x^2}}{\sqrt{x - 2}}$ is defined if $9 - x^2 \geq 0$ and $x - 2 > 0$

or $x \in [-3, 3]$ and $x > 2$ or $x \in (2, 3]$

$$g(x) = \sqrt{\frac{9 - x^2}{x - 2}} \text{ is defined if } \frac{9 - x^2}{x - 2} \geq 0 \text{ or}$$

$$\frac{x^2 - 9}{x - 2} \leq 0 \quad \begin{array}{c} - \quad + \quad - \quad + \\ \hline -3 \quad 2 \quad 3 \end{array}$$

From the sign scheme, $x \in (-\infty, -3] \cup (2, 3]$

Hence, $f(x)$ and $g(x)$ are identical if $x \in (2, 3]$

DPP 7

Total Marks 34

Time 60 Minute

1. Question Number 1, 2, 3, 4, 7, 8. **Marking Scheme** : +3 for correct answer 0 in all other cases. [(6.3)=18]
2. Question Number 5, 6. **Marking Scheme** : +8 for correct answer 0 in all other cases. [(2.8)=16]

1. Find the equivalent definition of $f(x) = \max. \{x^2, (1 - x)^2, 2x(1 - x)\}$ where $0 \leq x \leq 1$
2. Find the domain and range of $f(x) = \log \{x\}$, (where $\{ \}$)

represents the fractional part function).

3. Find the range of $f(x) = [\sin \{x\}]$, where $\{.\}$ represents the fractional part function and $[.]$ represents the greatest integer function.

4. Find the range of $f(x) = \frac{x - [x]}{1 - [x] + x}$, where $[.]$ represents the greatest integer function.

5. Find the domains of definitions of the following functions (Read the symbols $[*]$ and $\{*\}$ as greatest integers and fractional part functions respectively.)

(i) $f(x) = \frac{[x]}{2x - [x]}$

(ii) $f(x) = \frac{1}{[x]} + \log_{1 - \{x\}}(x^2 - 3x + 10) + \frac{1}{\sqrt{2 - |x|}} + \frac{1}{\sqrt{\sec(\sin x)}}$

6. $f(x) = \sqrt{\{x\}(x-1)(x-2)}$ Then find domain of $f(x)$. (Read the symbol $\{*\}$ as fractional part function)

7. Find range of $\text{Sgn}(\cos(2 \sin x))$.

Result Analysis

- 20 to 28 Marks : **Advanced Level.**
- 16 to 20 Marks : **Mains Level.**
- < 16 Marks : **Please go through above article again.**

8 CLASSIFICATION OF FUNCTION

8.1 ONE-ONE FUNCTION (INJECTION)

A function $f : A \rightarrow B$ is said to be a one-one function or an injection if different elements of A have different images in B.

Thus, $f : A \rightarrow B$ is one-one

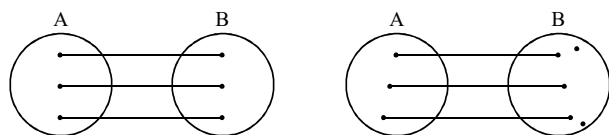
$$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b) \Rightarrow a = b \text{ for all } a, b \in A$$

For Example, $\mathbb{R} \rightarrow \mathbb{R}$ $f(x) = x^3 + 1$; $f(x) = e^{-x}$; $f(x) = \ln x$

Remember that a linear function is always one-one

Diagrammatically an injective mapping can be show as

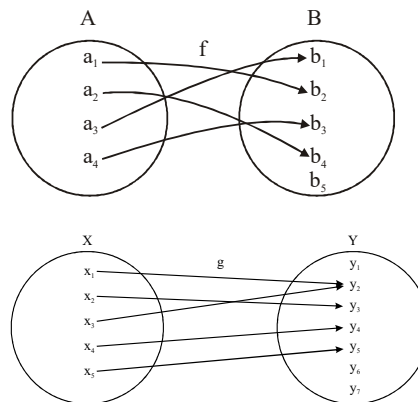


For Example, A function which associates to each country in the world, its capital, is one-one because different countries have their different capitals.

For Example, Let $f : A \rightarrow B$ and $g : X \rightarrow Y$ be two functions represented by the following diagrams :

Clearly, $f : A \rightarrow B$ is a one-one functions. But $g : X \rightarrow Y$ is

not one-one because two distinct elements x_1 and x_3 have the same image under function g .



For Example, Let $A = \{1, 2, 3, 4\}$ $B = \{1, 2, 3, 4, 5, 6\}$ and $f : A \rightarrow B$ be a function defined by $f(x) = x + 2$ for all $x \in A$.

We have, $f = \{(1, 3), (2, 4), (3, 5), (4, 6)\}$

Clearly, different elements in A have different images under function : So, $f : A \rightarrow B$ is a injection

Let $f : A \rightarrow B$ be a function such that A is an infinite set and we wish to check the injectivity of f. In such a case it is not possible to list the images of all elements of set A to see whether different elements of A have different images or not. The following algorithm provides a systematic procedure to check the injectivity of a function.

Method of Checking Injectivity of Function :

Method 1:

Step 1 Take two arbitrary elements x, y (say) in the domain of f .

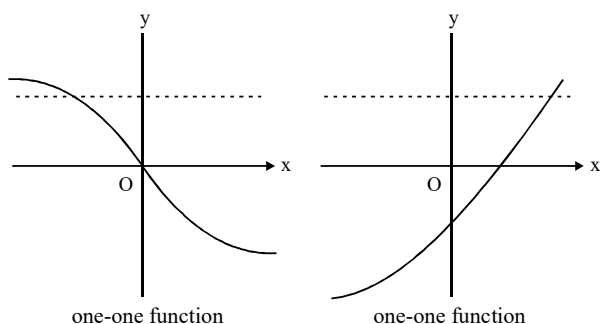
Step 2 Put $f(x) = f(y)$

Step 3 Solve $f(x) = f(y)$.

If $f(x) = f(y)$ gives $x = y$ only, then $f : A \rightarrow B$ is a one-one function (or an injection). Otherwise it is not.

Method 2:

	Description	Equivalent to number of ways in which n different balls can be distributed among r persons if	Number of functions
1	Total number of functions	Any one can get any number of object	r^n
2	Total number of one-to-one function	Each gets exactly 1 object or permutation of n different objects taken r at a time	$\begin{cases} {}^r C_n \cdot n!, & r \geq n \\ 0, & r < n \end{cases}$
3	Total number of many-one function	At least one gets more than one ball	$\begin{cases} r^n - {}^n C_1 n! & r \geq n \\ r^n, & r < n \end{cases}$
4	Total number of onto function	Each gets at least one ball	$\begin{cases} r^n - {}^r C_1 (r-1)^n + {}^r C_2 (r-2)^n - {}^r C_3 (r-3)^n + \dots, & r < n \\ r!, & r = n \\ 0, & r > n \end{cases}$
5	Total number of into function		$\begin{cases} {}^r C_1 (r-1)^n - {}^n C_2 (r-2)^n + {}^r C_3 (r-3)^n - \dots, & r \leq n \\ r^n, & r > n \end{cases}$
6	Total number of constant function	All the balls are received by any one person	r



Injectivity of a function can also be checked from its graph. If any straight line parallel to x-axis intersects the curve $y = f(x)$ exactly at one point, then the function $f(x)$ is one-one or an injection. Otherwise it is not.

Method 3:

If $f: R \rightarrow R$ is an injective map, then the graph of $y = f(x)$ is either a strictly increasing curve or a strictly decreasing curve. Consequently,

$$\frac{dy}{dx} > 0 \text{ or } \frac{dy}{dx} < 0 \text{ for all } x.$$

Number of ONE-ONE Function :

If A and B are finite sets having m and n elements respectively, then

Number of one-one functions from A to B is the number of arrangements of n items by taking m at a time i.e., ${}^n C_m \times m!$

Thus,

Number of one-one function from

$$A \text{ to } B = \begin{cases} {}^n C_m \times m!, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$$

8.2 MANY-ONE FUNCTION

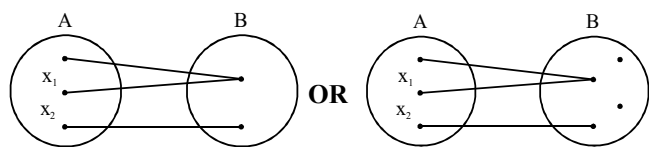
A function $f: A \rightarrow B$ is said to be a many-one function if two or more elements of set A have the same image in B .

Thus, $f: A \rightarrow B$ is a many-one function if there exists $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$.

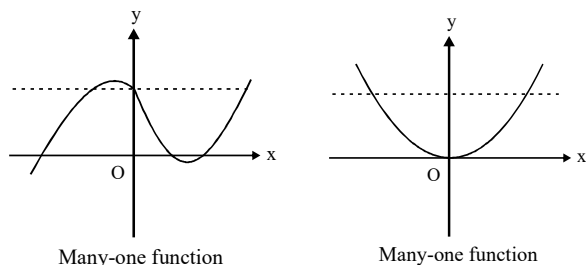
In other words, $f: A \rightarrow B$ is a many-one function if it is not a one-one function.

$$\text{No of Many one function} = \begin{cases} n^m - {}^m C_m \times m!, & n \geq m \\ n^m, & n < m \end{cases}$$

Example $R \rightarrow R$ $f(x) = [x]$; $f(x) = |x|$; $f(x) = ax^2 + bx + c$; $f(x) = \sin x$
Diagrammatically a many one mapping can be show as



Graphical Method to check whether function is many-one or not

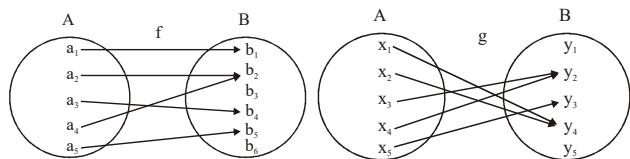


If a line parallel to x-axis, cuts the graph of the function atleast at two points, then f is many-one.

Note :

- (i) Any continuous function which has atleast one local maximum or local minimum in its domain, then $f(x)$ is Many-one.
- (ii) If a function is one-one, it cannot be many-one and vice versa. One One + Many One = Total number of mapping

Example 90 Let $f : A \rightarrow B$ and $g : X \rightarrow Y$ be two functions represented by the following diagrams :



Clearly $a_2 \neq a_4$ but $f(a_2) = f(a_4)$ and $x_1 \neq x_2$ but $g(x_1) = g(x_2)$. So, f and g are many-one functions.

Example 91 Find whether the following functions are one-one or not :

- (i) $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^3 + 2$ for all $x \in \mathbb{R}$
- (ii) $f : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(x) = x^2 + 1$ for all $x \in \mathbb{R}$

Solution Method 1

- (i) Let x, y be two arbitrary elements of \mathbb{R} (domain of f) such that $f(x) = f(y)$.

Then,

$$f(x) = f(y) \Rightarrow x^3 + 2 = y^3 + 2 \Rightarrow x^3 = y^3 \Rightarrow x = y$$

Hence, f is a one-one function from \mathbb{R} to itself.

- (ii) Let x, y be two arbitrary elements of \mathbb{Z} such that $f(x) = f(y)$.

Then,

$$f(x) = f(y) \Rightarrow x^2 + 1 = y^2 + 1 \Rightarrow x^2 = y^2 \Rightarrow x = \pm y$$

Here, $f(x) = f(y)$ does not provide the unique solution $x = y$ but it provides $x = \pm y$. So, f is not a one-one function. Infact, $f(2) = 2^2 + 1 = 5$ and $f(-2)^2 + 1 = 5$. So, 2 and -2 are

two distinct elements having the same image.

Method 2

(i) $f(x) = x^3 + 2$

$$f'(x) = 3x^2$$

$$f'(x) = 3x^2 \geq 0 \forall x \in \mathbb{R}$$

so function is one-one

(ii) $f(x) = x^2 + 1$

$$f'(x) = 2x$$

$f'(x)$ can be positive, negative or zero as $x \in \mathbb{R}$

so function is many-one

Example 92 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = \frac{x^2 + 4x + 7}{x^2 + x + 1}$. Is $f(x)$

one-one ?

Solution Method 1:

Let $f(x_1) = f(x_2)$ i.e.,

$$\frac{x_1^2 + 4x_1 + 7}{x_1^2 + x_1 + 1} = \frac{x_2^2 + 4x_2 + 7}{x_2^2 + x_2 + 1}$$

or $(x_1 - x_2)(2x_1 + 2x_2 + x_1x_2) = 0$

One solution is obviously $x_1 = x_2$.

Let us consider $2x_1 + 2x_2 + x_1x_2 + 1 = 0$

Here, we have got a relation between x_1 and x_2 and for each value of x_1 in the domain, we get a corresponding value of x_2 which may or may not be, the same as x_1 . Let us check this out :

If $x_1 = 0$, we get $x_2 = -1/2 \neq x_1$ and both lie in the domain of f.

Hence, we have two different values, $x_1 = 0$ and $x_2 = -1/2$, for which f(x) has the same value, that is $f(0) = f(-1/2) = 7$.

Hence, f is many-one.

Method 2

$$f(x) = \frac{x^2 + 4x + 7}{x^2 + x + 1}$$

$f : \mathbb{R} \rightarrow \mathbb{R}$ is an injective map

$$f'(x) = \frac{(x^2 + x + 1)(2x + 4) - (x^2 + 4x + 7)(2x + 1)}{(x^2 + x + 1)^2}$$

$$= \frac{2x^3 + 6x^2 + 6x + 4 - (2x^3 + 9x^2 + 18x + 7)}{(x^2 + x + 1)^2}$$

$$= -\frac{(3x^2 + 12x + 3)}{(x^2 + x + 1)^2}$$

$f'(x)$ can be positive, negative and zero, so function is Many-one

8.3 ONTO FUNCTION (SURJECTION)

A function $f : A \rightarrow B$ is said to be an onto function or a surjection if every element of B is the f -image of some element of A i.e., if $f(A) = B$, or range of f is the co-domain of f .

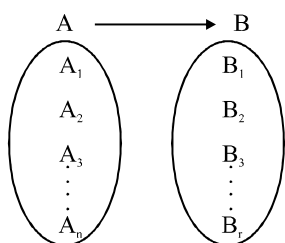
Thus, $f : A \rightarrow B$ is a surjection iff for $b \in B, \exists a \in A$ each such that $f(a) = b$.

NUMBER OF ONTO FUNCTIONS

If A and B are two sets having m and n elements respectively such that $1 \leq n \leq m$, then number of onto functions from A to B is given by

$$\sum_{r=1}^n (-1)^{n-r} {}^n C_r r^m$$

Derivation : Let set A contains 'n' elements and set B contains 'y' elements where $n \geq y$.



Number of onto function from $A \rightarrow B =$ Total number of function from $A \rightarrow B -$ Total number of into function from $A \rightarrow B$ (1)

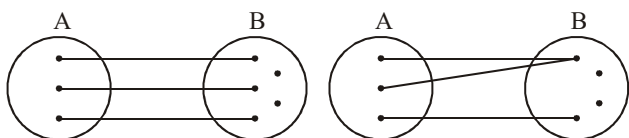
Total number of function = $r \times r \times r \times \dots n$ times (2)
[\because Every element is having r option]

8.4 INTOFUNCTION

A function $f : A \rightarrow B$ is an into function if there exists an element in B having no pre-image in A .

In other words, $f : A \rightarrow B$ is an into function if it is not an onto function.

e.g. $f : \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = [x], |x|, \text{sgn } x, f(x) = ax^2 + bx + c$



Methods to Determine Whether a Function is Onto or Into

1. If range = co-domain, then f is onto. If range is proper subset of co-domain, then f is into.
2. Solve $f(x) = y$ for x , say $x = g(y)$.
Now, if $g(y)$ is defined for each $y \in$ co-domain and $g(y) \in$ domain of f for all $y \in$ co-domain, then $f(x)$ is onto. If this requirement is not met by at least one value of y in the co-domain, then $f(x)$ is into.

NOTE :

1. An into function can be made onto by redefining the co-domain as the range of the original function.
2. Any polynomial function $f : \mathbb{R} \rightarrow \mathbb{R}$ is onto if the degree

of f is odd and into if the degree of f is even.

TOTAL NUMBER OF INTO FUNCTION

$$\sum_n (B_1 \cup B_2 \cup B_3 \cup B_4 \dots \cup B_r) = \sum_n (B_i) - \sum_n (B_i \cap B_j) + \sum_n (B_i \cap B_j \cap B_k) \dots + (-1)^n \sum_n (B_i \cap B_j \cap B_k)$$

where,

$\sum_n (B_i) =$ number of function from $A \rightarrow B$, when i^{th} elements is not being mapped from A

i.e., i^{th} element is not range of $f : A \rightarrow B$

$\sum_n (B_i) =$ Selection of that i^{th} element \times no. of possible function when i^{th} element is not in range of

$f : A \rightarrow B$

$$\sum_n (B_i) = {}^r C_1 \times (r-1)^n$$

$\sum_n (B_i \cap B_j) =$ Number of function from $A \rightarrow B$ when i^{th} and j^{th} is not being Mapped from A

i.e., i^{th} and j^{th} is not in Range of $f : A \rightarrow B$

$\sum_n (B_i \cap B_j) =$ Selection of those two i^{th} and j^{th} element \times number of possible function when i^{th} and j^{th} element is not in Range of $f : A \rightarrow B$

$$\sum_n (B_i \cap B_j) = {}^r C_2 \times (r-2)^n$$

Similarly, $\sum_n (B_i \cap B_j \cap B_k) = {}^r C_3 \times (r-3)^n$

Total number of into function(3)

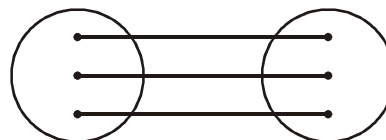
$$= {}^r C_1 (r-1)^n - {}^r C_2 (r-2)^n + {}^r C_3 (r-3)^n \dots + (-1)^{r-1} {}^r C_{r-1} \times 1$$

from (1), (2) and (3)
Number of onto function = $r^n - [{}^r C_1 (r-1)^n - {}^r C_2 (r-2)^n + {}^r C_3 (r-3)^n \dots + (-1)^{r-1} \times {}^r C_{r-1} \times 1]$

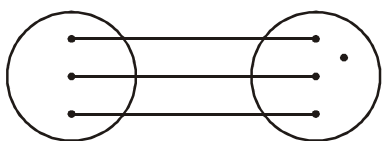
Number of onto function = $r^n - {}^r C_1 (r-1)^n + {}^r C_2 (r-2)^n - {}^r C_3 (r-3)^n \dots + (-1)^{r-1} \times {}^r C_{r-1}$

8.6 NUMBER OF FUNCTION (MAPPING)

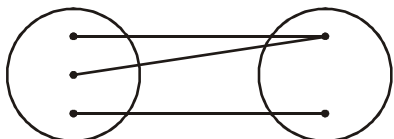
Thus A Function Can Be One Of These Four Types
one-one onto (injective & surjective)



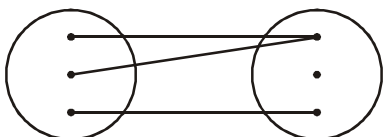
one-one into (injective but not surjective)



many-one onto (surjective but not injective)



many-one into (neither surjective nor injective)



NOTE :

1. If 'f' is both injective & surjective, then it is called a **Bijjective** mapping. The bijjective functions are also named as invertible, non singular or biuniform functions.
2. If a set A contains n distinct elements then the number of different functions defined from $A \rightarrow A$ is n^n & out of it $n!$ are one one and rest are many one.
3. $f: R \rightarrow R$ is a polynomial
 - (a) **Of even degree**, then it will neither be injective nor surjective.
 - (b) **Of odd degree**, then it will always be surjective, no general comment can be given on its injectivity.

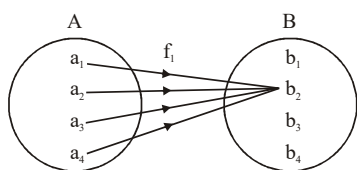
8.5 CONSTANTFUNCTION

$f: A \rightarrow B$ is said to be constant functions if every element of A has the same f image in B. Thus, $f: A \rightarrow B; f(x) = c, \forall x \in A, c \in B$ is constant function. Note that the range of a constant function is a singleton set

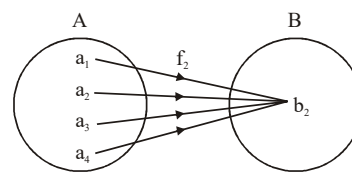
Domain : R

Range : {c}

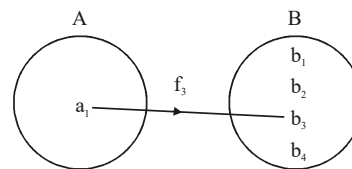
It should be noted that the range of a constant function contains only one element. Moreover, a constant function may be one-one or many-one, onto or into as shown in the following diagrams.



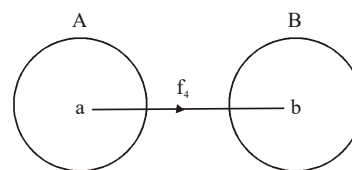
f_1 is many-one and into function.



f_2 is many-one and onto function.



f_3 is one-one and into function.



f_4 is one-one and onto function.

Example 93 Let $f: R \rightarrow R$, where $f(x) = \sin x$. Show that f is into.

Solution Since the co-domain of f is the set R, whereas the range of f is the interval $[-1, 1]$, f is into.

Can you make it onto ?

The answer is "yes", if you redefine the co-domain. Let f be defined from R to another set $Y = [-1, 1]$, i.e., $f: R \rightarrow Y$, where $f(x) = \sin x$. Then f is onto as the range of $f(x)$ is $[-1, 1] = Y$.

Example 94 Let $f: N \rightarrow Z$ be a function defined as $f(x) = x - 1000$. Show that f is an into function.

Solution Let $f(x) = y = x - 1000$

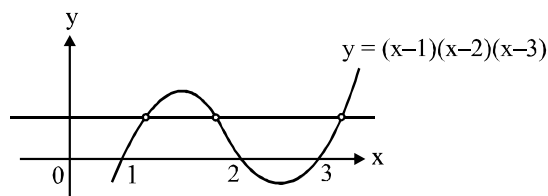
or $x = y + 1000 = g(y)$ (say)

Here, $g(y)$ is defined for each $y \in I$, but $g(y) \notin N$ for $y \leq -1000$. Hence, f is into.

Example 95 Show that $f: R \rightarrow R$ defined by $f(x) = (x - 1)(x - 2)(x - 3)$ is surjective but not injective.

Solution

$$f(1) = f(2) = f(3) = 0$$



As we can see a line parallel to x-axis is cutting graph at three points, So function is many-one

Example 96 If the function $f: R \rightarrow A$ given by $f(x) = \frac{x^2}{x^2 + 1}$ is

surjection, then find A.

Solution The domain of $f(x)$ is all real numbers.

Since, $f: \mathbb{R} \rightarrow A$ is surjective, A must be the range of $f(x)$.

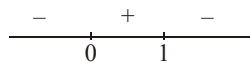
Let $f(x) = y$,

$$\text{i.e., } y = \frac{x^2}{x^2 + 1}$$

$$\text{or } x^2 y + y = x^2$$

$$\text{or } x = \sqrt{\frac{y}{1-y}}$$

$$\text{exists if } \frac{y}{1-y} \geq 0$$

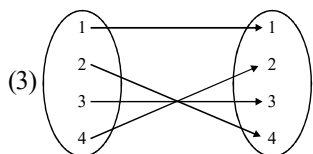
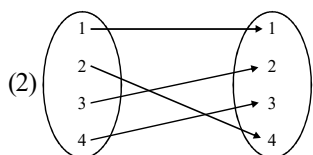
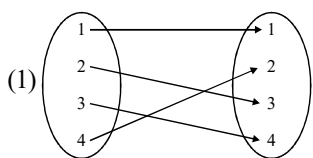


$$\text{or } 0 \leq y < 1$$

Hence, $A \in [0, 1)$

Example 97 If $A = \{1, 2, 3, 4\}$ and $f: A \rightarrow A$, then total number of invertible function 'f' such that $f(2) \neq 2, f(4) \neq 4, f(1) = 1$ is equal to

Solution Three possibilities are here.



Total no. of invertible function = 3

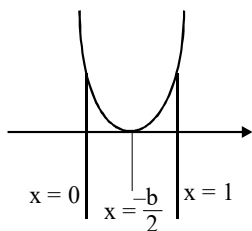
Example 98 $f(x) = x^2 + bx + 3$ is not injective for $x \in [0, 1]$, find value of b ?

Solution $f'(x) = 2x + b = 0$

$$x = \frac{-b}{2}$$

$$0 < \frac{-b}{2} < 1$$

$$b \in (-2, 0)$$



Example 99 Match the columns :

Column I

Column II

(A) $f(x) = \cos\left(\frac{\pi}{\sqrt{3}} \sin x + \sqrt{\frac{2}{3}} \pi \cos x\right)$ (P) Domain of $f(x)$

is $(-\infty, \infty)$

(B) $f(x) = \log_2(|\sin x| + 1)$

(Q) Range of $f(x)$

contains only one positive integer

(C) $f(x) = \cos\{[x] + [-x]\}$

(R) $f(x)$ is many-one function

(D) $f(x) = \lfloor \{e^x\} \rfloor$

(S) $f(x)$ is constant function

where $[x]$ and $\{x\}$ denotes greatest integer and fractional part function respectively.

Solution (A) \rightarrow P, Q, R ; (B) \rightarrow P, Q, R ; (C) \rightarrow P, Q, R, S ; (D) \rightarrow P, R, S

(A) $f(x) = \cos\left(\frac{\pi}{\sqrt{3}} \sin x + \sqrt{\frac{2}{3}} \pi \cos x\right)$

$$= \cos\left(\pi\left(\frac{1}{\sqrt{3}} \sin x + \sqrt{\frac{2}{3}} \cos x\right)\right)$$

$$= \cos\left(\pi \underbrace{(\sin(x+Q))}_{-1 \text{ to } 1}\right)$$

$$= \cos(-\pi \text{ to } \pi) \Rightarrow (-1 \text{ to } 1)$$

Domain = \mathbb{R}

function = Many to one

(B) $f(x) = \log_2\left(\underbrace{|\sin x|}_{0 \text{ to } 1} + 1\right)$

$$= \log_2(1 \text{ to } 2)$$

$$= [0, 1]$$

Domain = \mathbb{R}

function = Many-one because of $\sin x$

(C) $\cos\left(\underbrace{[x] + [-x]}_1\right)$

$$= \cos\{1\}$$

$$= \cos 0 = 1 \text{ always}$$

(D) $f(x) = \lfloor \{e^x\} \rfloor = \text{always } 0$

Example 100 Find the number of surjections from

$A = \{1, 2, \dots, n\}$ to $B = \{a, b\}$ for $n \geq 2$.

Solution Clearly, number of surjections from A to B .

$$= \sum_{r=1}^n (-1)^{2-r} {}^n C_r r^n$$

$$= (-1)^n {}^n C_1 (1)^n + (-1)^0 {}^n C_2 \cdot 2^n$$

$$= 2^n - 2$$

Example 101 Let $f: X \rightarrow Y$ be a function defined by $f(x) = a \sin(x + \pi/4) + b \cos x + c$. If f is a bijection, find the sets X and Y .

Solution : We have,

$$f(x) = a \sin(x + \pi/4) + b \cos x + c$$

$$\Rightarrow f(x) = \frac{a \sin x}{\sqrt{2}} + \left(\frac{a}{\sqrt{2}} + b\right) \cos x + c$$

$$\Rightarrow f(x) = r \sin(x + \theta) + c, \text{ where}$$

$$r = \sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{\sqrt{2}} + b\right)^2} \quad \text{and}$$

$$\theta = \tan^{-1}\left(\frac{a + \sqrt{2}b}{a}\right)$$

For f to be one-one, we must have

$$-\frac{\pi}{2} \leq x + \theta \leq \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{2} - \theta \leq x \leq \frac{\pi}{2} - \theta$$

$$\Rightarrow x \in \left[-\frac{\pi}{2} - \theta, \frac{\pi}{2} - \theta\right]$$

Hence, $X = \left[-\frac{\pi}{2} - \theta, \frac{\pi}{2} - \theta\right]$, where

$$\theta = \tan^{-1}\left(\frac{a + \sqrt{2}b}{a}\right)$$

For f to be onto, we must have

Range of $f = r$

Range of $f = [c - r, c + r]$ As $-1 \leq \sin(x + Q) \leq 1$

8.7 PERMUTATION AND COMBINATION PROBLEMS

Example 102 A function $f: A \rightarrow B$, such that set “A” and “B” contain four elements each then find

- (i) Total number of functions
- (ii) Number of one-one functions
- (iii) Number of many one functions
- (iv) Number of onto functions
- (v) Number of into functions

Solution

- (i) 1st element of A can have its image in 4 ways. Similarly II, III and IV can have 4 options for their image each.
Hence number of functions = 4^4
- (ii) 4 different elements can be matched in $4!$ ways
- (iii) Number of many one functions
= Total number of functions – number of one-one functions
= $4^4 - 4!$

(iv) Since 4 elements in B are given hence each should be image of atleast one.

So number of onto function = $4!$

(v) Number of into functions = $4^4 - 4!$

Example 103 A function $f: A \rightarrow B$, such that set “A” contains five element and “B” contains four elements then find

- (i) Total number of functions
- (ii) Number of one-one functions
- (iii) Number of onto functions
- (iv) Number of many-one function
- (v) Number of into functions

Solution

- (i) Total number of functions
Hence number of functions = $4 \times 4 \times 4 \times 4 \times 4 = 4^5$
- (ii) Number of one-one functions
Since A contains five elements hence one-one function is not possible.
- (iii) Number of onto function
= $4^5 - {}^4C_1 3^5 + {}^4C_2 2^5 - {}^4C_3 1^5$
= $1024 - 972 + 192 - 4$
= 240
- (iv) Number of many one function
All the possible functions are many-one
= $4^5 = 1024$
- (v) Number of into functions
Number of into function = Total number of functions – number of onto functions
= $1024 - 240 = 784$

Example 104 A function $f: A \rightarrow B$ such that set A contains 4 elements and set B contains 5 elements, then find the

- (i) Total number of functions
- (ii) Number of injective (one-one) mapping
- (iii) Number of many-one function
- (iv) Number of onto function
- (v) Number of into functions

Solution

- (i) Total number of functions
Every element in 4 has 5 options for image,
Hence, total number of functions $5^4 = 625$
- (ii) Number of injective (one-one) mapping.
4 elements in A needs four images
Hence, number of one-one functions = ${}^5C_4 \times 4! = 120$
- (iii) Number of many-one functions
Number of many-one mapping
= Total number of mapping – number of one-one mapping
= $5^4 - {}^5C_4 \times 4! = 505$
- (iv) Number of onto function = 0
- (v) Number of into functions = $5^4 = 625$

Example 105 For a real number x , let $[x]$ denote the greatest integer less than or equal to x .

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = 2x + [x] + \sin x \cdot \cos x$ then

f is

- (A) one-one but not onto
- (B) onto but not one-one
- (C) both one-one and onto
- (D) neither one-one nor onto

Solution (A)

If $x = a$, where 'a' is an integer then $f(a) = 2a + a + \frac{1}{2} \sin 2a$

But $\lim_{h \rightarrow 0} f(a-h) = 2a + (a-1) + \frac{1}{2} \sin 2a$

Values between $\lim_{h \rightarrow 0} f(a-h)$ and $f(a)$ are never achieved.

Also, $f'(x) = 2 + \cos 2x > 0$, i.e., $f(x)$ is strictly increasing.

Example 106 If the following functions are defined from \mathbb{R} to \mathbb{R} then identify the function which is bijective ?

- (A) $f(x) = \frac{e^x + e^{-x}}{2}$
- (B) $f(x) = x^4 - 3x^3 + 1$
- (C) $f(x) = 18x^3 - 21x^2 + 8x - 1$
- (D) $f(x) = x^3 - 4x^2 + 16x + 17$

Solution (D)

(A) $f(x) = \frac{e^x + e^{-x}}{2}$ is even function

(B) $f(x) = x^4 - 3x^3 + 1$ is an even degree polynomial function

(C) $f(x) = 18x^3 - 21x^2 + 8x - 1$

$f(x)$ is an odd degree polynomial function hence its range is \mathbb{R} .

$f'(x) = 54x^2 - 42x + 8$

$\Rightarrow D > 0$, hence, it is many one function

(D) $f(x) = x^3 - 4x^2 + 16x + 17$

$f(x)$ is an odd degree polynomial function hence its range \mathbb{R} .

$f'(x) = 3x^2 - 8x + 16$

$\Rightarrow D < 0$, hence $f(x)$ is one-one as well as onto

Example 107 If the function $f(x)$ and $g(x)$ are defined on $\mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) = \begin{cases} x+3, & x \in \text{rational} \\ 4x, & x \in \text{irrational} \end{cases} \text{ and}$$

$$g(x) = \begin{cases} x + \sqrt{5}, & x \in \text{irrational} \\ -x, & x \in \text{rational} \end{cases}$$

then $(f-g)(x)$ is

- (A) one-one and onto
- (B) neither one-one nor onto
- (C) one-one but not onto
- (D) onto but not one-one

Solution (B)

We have, $(f-g)(x) = (f(x) - g(x))$

$$= \begin{cases} 2x+3, & x \in \text{rational} \\ 3x-\sqrt{5}, & x \in \text{irrational} \end{cases}$$

As, $f\left(\frac{-3}{2}\right) = 0 = f\left(\frac{\sqrt{5}}{3}\right)$

and so on.

$\Rightarrow f(x)$ is many one function

Also $-\sqrt{5}$ does not belong to the range, because if

$$3x - \sqrt{5} = -\sqrt{5}$$

$\therefore x = 0 \notin \mathbb{Q}^c$

$\Rightarrow f(x)$ is into function

DPP 8

Total Marks 38

Time 30 Minute

1. Question Number 1, 2, 3, 4, 5, 10. **Marking Scheme** : +3 for correct answer -1 in all other cases. [(6.3)=18]
2. Question Number 6. **Marking Scheme** : +2 for correct answer 0 in all other cases. [(3.2)=6]
3. Question Number 7 . **Marking Scheme** : +2 for correct answer 0 in all other cases. [(4.2)=8]
4. Question Number 8, 9. **Marking Scheme** : +3 for correct answer -1 in all other cases. [(2.3)=6]

1. Let $A = \{-1, 1, -2, 2\}$ and $B = \{1, 4, 9, 16\}$, $f(x) = x^2$ then prove that $f: A \rightarrow B$ is many-one.
2. Consider a function $f: \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(x) = |x|$. Deetermine whether f is one-one or many-one.
3. If $f: X \rightarrow [1, \infty)$ is a function defined as $f(x) = 1 + 3x^3$, find the super-set of all the sets X such that $f(x)$ is one-one.
4. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $f(x) = x^3 + x^2 + 3x + \sin x$, then find whether function one-one or onto ?
5. Let $A = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\} = B$. Show that $f: A \rightarrow B$ given by $f(x) = x|x|$ is a bijection.
6. Classify the following functions $f(x)$ defined in $\mathbb{R} \rightarrow \mathbb{R}$ as injective, surjective, both or none.
 - (i) $f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18}$
 - (ii) $f(x) = x^3 - 6x^2 + 11x - 6$
 - (iii) $f(x) = (x^2 + x + 5)(x^2 + x - 3)$
7. Let $A = \{x: -1 \leq x \leq 1\} = B$ be a mapping $f: A \rightarrow B$. Then, match the following columns :

Column I (Function)	Column II (Type of mapping)
p. $f(x) = x $	a. one-one
q. $f(x) = x x $	b. many-one
r. $f(x) = x^3$	c. onto
s. $f(x) = [x]$, where $[.]$ represents greatest integer function	d. into

t. $f(x) = \sin \frac{\pi x}{2}$

8. If $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \begin{cases} x|x| - 4 & x \in \mathbb{Q} \\ x|x| - \sqrt{3} & x \notin \mathbb{Q} \end{cases}$, then identify the type of function.
9. If $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4, 5, 6\}$ and $f: A \rightarrow B$ is an injective mapping satisfying $f(i) \neq i$, then number of such mappings are :
 (A) 182 (B) 181
 (C) 183 (D) None of these
10. Let $A = \{1, 2, 3, 4\}$ and $B = \{0, 1, 2, 3, 4, 5\}$. Find the number of onto function $g: B \rightarrow A$?

Result Analysis

- 30 to 38 Marks : **Advanced Level.**
- 19 to 30 Marks : **Mains Level.**
- < 19 Marks : **Please go through above artical again.**

9. FUNCTIONALEQUATION

Functional equation is any equation that specifies a function in implicit form. Often, the equation relates the value of a function (or functions) at some point with its values at other points. For instance, the properties of functions can be determined by considering the types of functional equations they satisfy.

A. Functional Equations Satisfied by Typical Functions

- $f(x + y) = f(x) \cdot f(y)$ is satisfied by $f(x) = a^x$ as $f(x + y) = a^{x+y} = a^x a^y = f(x) \cdot f(y)$.
- $f(x - y) = \frac{f(x)}{f(y)}$ is satisfied by $f(x) = a^x$ as $f(x - y) = a^{x-y} = \frac{a^x}{a^y} = \frac{f(x)}{f(y)}$.
- $f(x) + f(y) = f(xy)$ is satisfied by $f(x) = \log_a x$ as $f(x) + f(y) = \log_a x + \log_a y = \log_a xy = f(xy)$.
- $f(x) - f(y) = f\left(\frac{x}{y}\right)$ is satisfied by $f(x) = \log_a x$ as $f(x) - f(y) = \log_a x - \log_a y = \log_a \frac{x}{y} = f\left(\frac{x}{y}\right)$.
- $f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ is satisfied by polynomial function $f(x) = \pm x^n + 1$

B. Functional Equations Resulting from Properties of Functions

- Odd functions having symmetry of graph about the origin :
 $f(x) + f(-x) = 0, \forall x \in D_f$
- Even functions having symmetry of graph about the y-axis :
 $f(x) = f(-x), \forall x \in D_f$
- Symmetry of graph about the point (a, 0) :
 $f(a - x) = -f(a + x)$
- Symmetry of graph about the line $x = a$: $f(a - x) = f(a + x)$

Example 108 Let a function $f(x)$ satisfied $f(x) + f(2x) + f(2 - x) + f(1 + x) = x \forall x \in \mathbb{R}$. Then find the value of $f(0)$.

Solution $f(x) + f(2x) + f(2 - x) + f(1 + x) = x$
 Put $x = 0$. Then
 $f(0) + f(0) + f(2) + f(1) = 0$
 or $2f(0) + f(1) + f(2) = 0$ (i)
 Put $x = 1$. Then,
 $f(1) + f(2) + f(1) + f(2) = 1$
 $\therefore f(1) + f(2) = 1/2$
 So, form (i)
 $2f(0) + 1/2 = 0$
 $\therefore f(0) = -1/4$

Example 109 Let $f(x) = x + f(x - 1)$ for $\forall x \in \mathbb{R}$. If $f(0) = 1$, find $f(100)$.

Solution Given $f(x) = x + f(x - 1)$ and $f(0) = 1$
 Put $x = 1$. Then,
 $f(1) = 1 + f(0) = 2$
 Put $x = 2$. Then,
 $f(2) = 2 + f(1) = 4$
 Put $x = 3$. Then,
 $f(3) = 3 + f(2) = 7$
 Thus, $f(0), f(1), f(2), \dots$ form a series 1, 2, 4, 7, ...
 Let $S = 1 + 2 + 4 + 7 + \dots + f(n - 1)$
 $S = 1 + 2 + 4 + \dots + f(n - 2) + f(n - 1)$
 Subtracting, we get
 $0 = (1 + 1 + 2 + 3 + \dots + n \text{ terms}) - f(n - 1)$
 $\therefore f(n - 1) = 1 + \frac{n(n - 1)}{2}$
 $\therefore f(100) = 5051$

Example 110 Let f be a function such that $f(3) = 1$ and $f(3x) = x + f(3x - 3)$ for all x . Then find the value of $f(300)$.

Solution $f(3) = 1$
 Put $x = 2$
 $f(6) = 2 + f(3)$
 $f(6) = 2 + 1 = 3$
 Put $x = 3$

$$f(9) = 3 + f(6) \\ = 3 + 3 = 6$$

Put $x=4$

$$f(12) = 4 + f(9) \\ = 4 + 6$$

$$f(12) = 10$$

$$f(300) = f(3 \times 100)$$

So, 100th term of series will be

$$S = 1 + 3 + 6 + 10 + 15 + \dots a_{100}$$

$$S = 1 + 3 + 6 + 10 + \dots a_{100}$$

$$a_{100} = 1 + 2 + 3 + 4 + \dots 100$$

$$a_{100} = \frac{100 \times 101}{2} = 5050 = f(300)$$

Example 111 Consider a real-valued function $f(x)$ satisfying

$$2f(xy) = (f(x))^y + (f(y))^x \quad \forall x, y \in \mathbb{R} \text{ and } f(1) = a, \text{ where}$$

$$a \neq 1. \text{ Prove that } (a-1) \sum_{i=1}^n f(i) = a^{n+1} - a$$

Solution We have, $2f(xy) = (f(x))^y + (f(y))^x$.

Replacing y by 1, we get

$$2f(x) = f(x) + (f(1))^x \text{ or } f(x) = a^x$$

$$\text{or } \sum_{i=1}^n f(i) = a + a^2 + \dots + a^n = \frac{a^{n+1} - a}{a - 1}$$

$$\text{or } (a-1) \sum_{i=1}^n f(i) = a^{n+1} - a$$

Example 112 Let f be a function satisfying of x . Then, $f(xy)$

$$= \frac{f(x)}{y} \text{ for all positive real numbers } x \text{ and } y. \text{ If } f(30) = 20,$$

then find the value of $f(40)$.

Solution Given $f(xy) = \frac{f(x)}{y}$

$$\text{or } f(y) = \frac{f(1)}{y} \quad (\text{putting } x = 1)$$

$$\text{or } f(30) = \frac{f(1)}{30}$$

$$\text{or } f(1) = 30 \times f(30) = 30 \times 20 = 600$$

$$\therefore f(40) = \frac{f(1)}{40} = \frac{600}{40} = 15$$

Example 113 If $f: \mathbb{R} \rightarrow \mathbb{R}$ is an odd function such that

a. $f(1+x) = 1 + f(x)$

b. $x^2 f\left(\frac{1}{x}\right) = f(x), x \neq 0$

then find $f(x)$.

Solution $x^2 f\left(\frac{1}{x}\right) = f(x), x \neq 0$

Replacing x by $x+1$. Then,

$$(x+1)^2 f\left(\frac{1}{1+x}\right) = f(x+1)$$

$$f\left(\frac{1}{1+x}\right) = \frac{1+f(x)}{(1+x)^2} \quad \dots(i)$$

$$\therefore f\left(\frac{1}{1+x}\right) = f\left(1 - \frac{x}{1+x}\right)$$

$$= 1 + f\left(-\frac{x}{1+x}\right)$$

$$= 1 - f\left(\frac{x}{1+x}\right) \quad (\because f(x) \text{ is odd})$$

$$= 1 - \left(\frac{x}{1+x}\right)^2 f\left(\frac{1+x}{x}\right)$$

$$= 1 - \left(\frac{x}{1+x}\right)^2 f\left(1 + \frac{1}{x}\right)$$

$$= 1 - \left(\frac{x}{1+x}\right)^2 \left(1 + \frac{f(x)}{x^2}\right) \quad \dots(ii)$$

From (i) and (ii),

$$\frac{1+f(x)}{(1+x)^2} = 1 - \left(\frac{x}{1+x}\right)^2 \left(1 + \frac{f(x)}{x^2}\right)$$

$$\text{or } 1 + f(x) = (1+x)^2 - x^2 - f(x)$$

$$\text{or } f(x) = x$$

Example 114 Let $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ be a function which satisfies

$$f(x) \cdot f(y) = f(xy) + 2 \left(\frac{1}{x} + \frac{1}{y} + 1\right) \text{ for } x, y > 0$$

Then find $f(x)$.

Solution Put $x = 1$ and $y = 1$.

Then,

$$\therefore f^2(1) - f(1) - 6 = 0$$

$$\text{i.e., } f(1) = 3 \text{ or } f(1) = -2$$

Now, put $y = 1$.

$$\text{Then, } f(x) \cdot f(1) = f(x) + 2 \left(\frac{1}{x} + 2\right) = f(x) + 2 \left(\frac{2x+1}{x}\right)$$

$$\text{or } f(x) [f(1) - 1] = \frac{2(2x+1)}{x}$$

or $f(x) = \frac{2(2x+1)}{x[f(1)-1]}$

For $f(1) = 3$,

$f(x) = \frac{2x+1}{x}$ and for $x = -2$.

$f(x) = \frac{2(2x+1)}{-3x}$

Example 115 If $f(x+y) = f(x) \cdot f(y)$ for all real x, y and $f(0) \neq 0$,

then prove that the function $g(x) = \frac{f(x)}{1 + \{f(x)\}^2}$

is an even function.

Solution : Given $f(x+y) = f(x) \cdot f(y)$. Put $x = y = 0$. Then $f(0) = 1$.

Put $y = -x$. Then

$f(0) = f(x)f(-x)$ or $f(-x) = \frac{1}{f(x)}$

Now, $g(x) = \frac{f(x)}{1 + \{f(x)\}^2}$

$g(-x) = \frac{f(-x)}{1 + \{f(-x)\}^2} = \frac{\frac{1}{f(x)}}{1 + \frac{1}{\{f(x)\}^2}} = \frac{f(x)}{1 + \{f(x)\}^2} = g(x)$

Example 116 If $f(x) = \frac{a^x}{a^x + \sqrt{a}}$ ($a > 0$), then find the value of

$\sum_{r=1}^{2n-1} 2f\left(\frac{r}{2n}\right)$.

Solution $f(x) = \frac{a^x}{a^x + \sqrt{a}}$

$\Rightarrow f(1-x) = \frac{a^{1-x}}{a^{1-x} + \sqrt{a}} = \frac{a^1}{a^1 + \sqrt{a}a^x} = \frac{\sqrt{a}}{\sqrt{a} + a^x}$

$\Rightarrow f(x) + f(1-x) = 1$

Also, $f\left(\frac{1}{2}\right) = \frac{1}{2}$

$\Rightarrow \sum_{r=1}^{2n-1} 2f\left(\frac{r}{2n}\right)$

$$\Rightarrow 2f\left[\left(\frac{1}{2n}\right) + f\left(\frac{2}{2n}\right) + \dots + f\left(\frac{n-1}{2n}\right) + f\left(\frac{n}{2n}\right) + f\left(\frac{n+1}{2n}\right) + \dots + f\left(\frac{2n-1}{2n}\right)\right]$$

$$\Rightarrow 2\left[\left[f\left(\frac{1}{2n}\right) + f\left(\frac{2n-1}{2n}\right)\right] + \left[f\left(\frac{2}{2n}\right) + f\left(\frac{2n-2}{2n}\right)\right] + \dots + \left[f\left(\frac{n-1}{2n}\right) + f\left(\frac{n+1}{2n}\right)\right] + f\left(\frac{1}{n}\right)\right]$$

$$\Rightarrow 2\left[\left[f\left(\frac{1}{2n}\right) + f\left(1 - \frac{1}{2n}\right)\right] + \left[f\left(\frac{2}{2n}\right) + f\left(1 - \frac{2}{2n}\right)\right] + \dots + \left[f\left(\frac{n-1}{2n}\right) + f\left(1 - \frac{n-1}{2n}\right)\right] + \frac{1}{2}\right]$$

$= 2[1 + 1 + 1 + \dots + (n-1) \text{ times}] + 1$
 $= 2n - 1$

DPP 9

Total Marks 30 **Time 30 Minute**

1. Question Number 1 to 10. **Marking Scheme :** +3 for correct answer 0 in all other cases. [(10.3)=30]

1. The function $f(x)$ is defined for all real x . If $f(a+b) = f(ab)$

$\forall a$ and b and $f\left(-\frac{1}{2}\right) = -\frac{1}{2}$, then find the value of $f(1005)$.

2. If $f(x)$ is a polynomial function satisfying

$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and $f(4) = 65$, then find $f(6)$.

3. Let f be a real-valued function such that

$f(x) + 2f\left(\frac{2002}{x}\right) = 3x$ Then find $f(x)$.

4. A continuous function $f(x)$ on $\mathbb{R} \rightarrow \mathbb{R}$ satisfies the relation $f(x) + f(2x+y) + 5xy = f(3x-y) + 2x^2 + 1$ for $\forall x, y \in \mathbb{R}$. Then find $f(x)$.

5. Prove that $f(x)$ given by $f(x+y) = f(x) + f(y) \forall x \in \mathbb{R}$ is an odd function.

6. Let $f(x) = \frac{9^x}{9^x + 3}$. Show $f(x) + f(1-x) = 1$ and, hence, evaluate

$$f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + f\left(\frac{3}{1996}\right) + \dots + f\left(\frac{1995}{1996}\right)$$

- A real valued function $f(x)$ satisfies the functional equation $f(x - y) = f(x) f(y) - f(a - x) f(a + y)$ where 'a' is a given constant and $f(0) = 1$, $f(2a - x)$ is equal to :
 (A) $-f(x)$ (B) $f(x)$
 (C) $f(a) + f(a - x)$ (D) $f(-x)$
- The function $f(x)$ satisfy the equation $f(1 - x) + 2f(x) = 3x \quad \forall x \in \mathbb{R}$, then $f(0) =$
 (A) -2 (B) -1
 (C) 0 (D) 1
- Suppose f is a real function satisfying $f(x + f(x)) = 4f(x)$ and $f(1) = 4$. Find the value of $f(21)$.
- $f(x) + 3x f\left(\frac{1}{x}\right) = 2(x + 1) \quad \forall x > 0$. Find $f(10099)$

Result Analysis

- 24 to 30 Marks : **Advanced Level.**
- 15 to 23 Marks : **Mains Level.**
- < 15 Marks : **Please go through above artical again.**

10. COMPOSITE FUNCTION

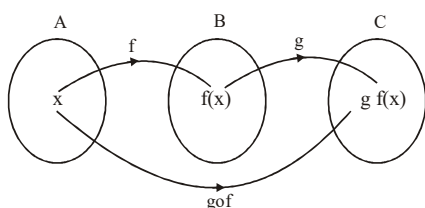
Let A, B and C be three non-empty sets.
 Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions. Then $g \circ f : A \rightarrow C$. This function is called the composition of f and g and is given by

$$g \circ f(x) = g(f(x)) \quad \forall x \in A$$

Thus, the image of every $x \in A$ under the function $g \circ f$ is the g -image of the f -image of x .

The $g \circ f$ is defined only if $\forall x \in A, f(x)$ is an element of the domain of g so that we can take its g -image.

The range of f must be a subset of the domain of g in $g \circ f$.



Properties of Composite Functions :

- The composition of functions is not commutative in general, i.e., $f \circ g \neq g \circ f$.
- The composition of functions is associative, i.e, if $h : A \rightarrow B, g : B \rightarrow C$, and $f : C \rightarrow D$ be three functions, then $(f \circ g) \circ h = f \circ (g \circ h)$.
- The composition of any function with the identity function is the function itself, i.e., if $f : A \rightarrow B$, then $f \circ I_A = I_B \circ f = f$, where I_A and I_B are the identity functions of A and B, respectively.

- If $f : A \rightarrow B$ and $g : B \rightarrow C$ are one-one, then $g \circ f : A \rightarrow C$ is also one-one.

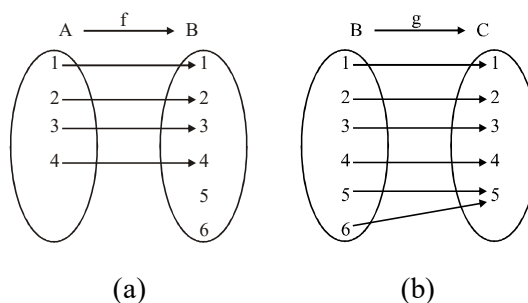
Proof Suppose $g \circ f(x_1) = g \circ f(x_2)$
 or $g(f(x_1)) = g(f(x_2))$
 or $f(x_1) = f(x_2)$ (As g is one-one)
 or $x_1 = x_2$ (As f is one-one)
 Hence, $g \circ f$ is one-one.

- If $f : A \rightarrow B$ and $g : B \rightarrow C$ are onto, then $g \circ f : A \rightarrow C$ is also onto.

Proof Given an arbitrary element $z \in C$, there exists a pre-image y of z under g such that $g(y) = z$, since g is onto. Further, for $y \in B$, there exists an element x in A with $f(x) = y$, since f is onto. Therefore, $g \circ f(x) = g(f(x)) = g(y) = z$, showing that $g \circ f$ is onto.

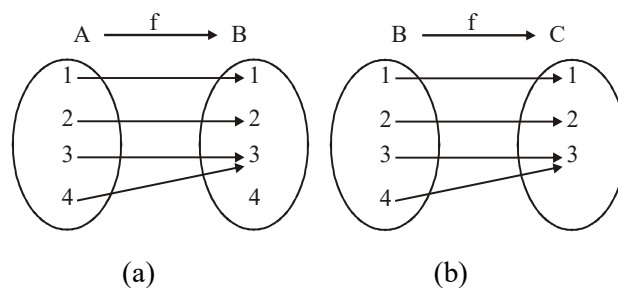
- If $g \circ f(x)$ is one-one, then $f(x)$ is necessarily one-one but $g(x)$ may not be one-one.

Consider the functions $f(x)$ and $g(x)$ as shown in the following figure.



Here, f is one-one, but g is many-one. But $g \circ f(x) : \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ is one-one.

- If $g \circ f(x)$ is onto, then $g(x)$ is necessarily onto but $f(x)$ may not be onto.



Here, f is into and g is onto. But $g \circ f(x) : \{(1, 1), (2, 2), (3, 3), (4, 3)\}$ is onto.

Thus, it can be verified in general that $g \circ f$ is one-one implies f is one-one. Similarly, $g \circ f$ is onto implies g is onto.

Example 117 Let $f : \{2, 3, 4, 5\} \rightarrow \{3, 4, 5, 9\}$ and $g : \{3, 4, 5, 9\} \rightarrow \{7, 11, 15\}$ be functions defined as $f(2) = 3, f(3) = 4, f(4) = 5, f(5) = 9, g(3) = 7, g(4) = 11, g(5) = 15$. Find $g \circ f$.

Solution we have $\text{gof}(2) = g(f(2)) = g(3) = 7$, $\text{gof}(3) = g(f(3)) = g(4) = 7$, $\text{gof}(4) = g(f(4)) = g(5) = 11$ and $\text{gof}(5) = g(5) = 11$.

Example 118 Let $f(x)$ and $g(x)$ be bijective functions where $f: \{a, b, c, d\} \rightarrow \{1, 2, 3, 4\}$ and $g: \{3, 4, 5, 6\} \rightarrow \{w, x, y, z\}$, respectively. Then, find the number of elements in the range of $g(f(x))$.

Solution The range of $f(x)$ for which $g(f(x))$ is defined as $\{3, 4\}$. Hence, the domain of $g\{f(x)\}$ has two elements. Therefore, the range of $g(f(x))$ also has two elements.

Example 119 Suppose that $g(x) = 1 + \sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x$. Then find the function $f(x)$.

Solution : $g(x) = 1 + \sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x \dots(i)$

$$\therefore f(1 + \sqrt{x}) = 3 + 2\sqrt{x} + x$$

Put $1 + \sqrt{x} = y$ or $x = (y - 1)^2$. Then,

$$f(y) = 3 + 2(y - 1) + (y - 1)^2 = 2 + y^2.$$

$$\therefore f(x) = 2 + x^2$$

Example 120 The function $f(x)$ is defined in $[0, 1]$. Find the domain of $f(\tan x)$.

Solution : Here, $f(x)$ is defined in $[0, 1]$

So, $x \in [0, 1]$, i.e., the only value of x that we can substitute lies in $[0, 1]$

For $f(\tan x)$ to be defined, we must have

$$0 \leq \tan x \leq 1 \quad [\text{As } x \text{ is replaced by } \tan x]$$

$$\text{i.e., } n\pi \leq x \leq n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$$

Thus, the domain of $f(\tan x)$ is

$$\left[n\pi, n\pi + \frac{\pi}{4} \right], n \in \mathbb{Z}$$

Example 121 $f(x) = \begin{cases} x+1, & x < 0 \\ x^2, & x \geq 0 \end{cases}$ and $g(x) = \begin{cases} x^3, & x < 1 \\ 2x-1, & x \geq 1 \end{cases}$

Then, find $f(g(x))$ and find its domain and range.

Solution In such kind of questions follow this general procedure.

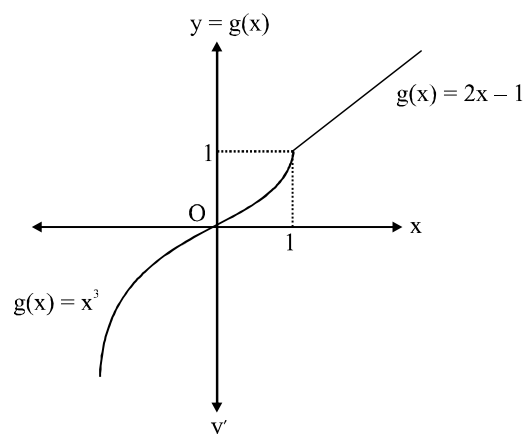
Step 1 : To find $f(g(x))$ replace 'x' by $g(x)$, in the definition of $f(x)$, here it becomes

$$f(x) = \begin{cases} x+1, & x < 0 \\ x^2, & x \geq 0 \end{cases} \quad x \rightarrow g(x)$$

$$f(g(x)) = \begin{cases} g(x)+1, & g(x) < 0 \\ (g(x))^2, & g(x) \geq 0 \end{cases}$$

Step 2 : Now draw graph of $g(x)$. If we have to find $g(f(x))$ then draw graph at $f(x)$.

$$\text{here, } g(x) = \begin{cases} x^3, & x < 1 \\ 2x-1, & x \geq 1 \end{cases}$$



Step 3 : Now see the inverse for which composite function is defined. Here we can see

$$f(g(x)) = \begin{cases} g(x)+1, & g(x) < 0 \\ (g(x))^2, & g(x) \geq 0 \end{cases}$$

Here the intervals are (i) $g(x) < 0$ (ii) $g(x) \geq 0$

Step 4 : (i) First interval is $g(x) < 0$. Now see graph of $g(x)$ in Step - 2, Now down all possible equation of $g(x)$ and corresponding values of 'x' for which $g(x) < 0$; there is only one equation i.e., $g(x) = x^3, x < 0$ Substitute this in $f(g(x))$ so it becomes

$$f(g(x)) = \begin{cases} g(x)+1, & g(x) < 0 \\ (g(x))^2, & g(x) \geq 0 \end{cases}$$

$$\Rightarrow f(g(x)) = \begin{cases} x^3 + 1, & x < 0 \\ (g(x))^2, & g(x) \geq 0 \end{cases}$$

(ii) Now see second interval i.e., $g(x) \geq 0$ and follow the same procedure (Step - 4). Now there are two possibilities for $g(x)$

$$g(x) = \begin{cases} x^3, & 0 \leq x < 1 \\ 2x-1, & x \geq 1 \end{cases}$$

Substitute this in $f(g(x))$

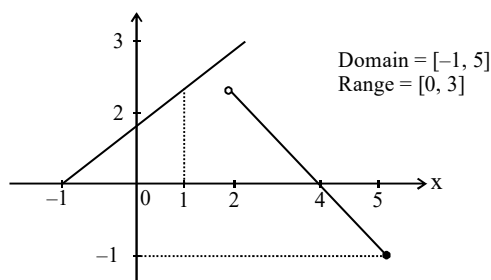
$$f(g(x)) = \begin{cases} x^3 + 1 & x < 0 \\ (x^3)^2 & 0 \leq x < 1 \\ (2x-1)^2 & x \geq 1 \end{cases}$$

$$f(g(x)) = \begin{cases} x^3 + 1 & x < 0 \\ x^6 & 0 \leq x < 1 \\ (2x-1)^2 & x \geq 1 \end{cases}$$

Example 122 $f(x) = \begin{cases} x+1, & -1 \leq x \leq 2 \\ 4-x, & 2 < x \leq 5 \end{cases}$. Find domain and range of $f(f(x))$.

Solution $f(x) = \begin{cases} 1+x & -1 \leq x \leq 2 \\ 4-x & 2 < x \leq 5 \end{cases}$

$f(f(x)) = \begin{cases} f(x)+1 & -1 \leq f(x) \leq 2 \\ 4-f(x) & 2 < f(x) \leq 5 \end{cases}$



$f(x) = \begin{cases} (x+1)+1 & -1 \leq x \leq 1 \\ 4-(x+1) & 1 \leq x \leq 2 \\ 4-x+1 & 2 < x \leq 5 \end{cases}$

$f \circ f = \begin{cases} x+2 & -1 \leq x \leq 1 \\ -x+3 & 1 \leq x \leq 2 \\ 5-x & 2 < x \leq 5 \end{cases}$

Example 123 Let $f(x) = \frac{x}{\sqrt{1+x^2}}$, then $\underbrace{f \circ f \circ f \dots \circ f}_{n \text{ times}}(x)$ is :

- (A) $\frac{x}{\sqrt{1 + \left(\sum_{r=1}^n r\right) x^2}}$ (B) $\frac{x}{\sqrt{1 + \left(\sum_{r=1}^n 1\right) x^2}}$
 (C) $\left(\frac{x}{\sqrt{1+x^2}}\right)^n$ (D) $\frac{nx}{\sqrt{1+nx^2}}$

Solution $f(x) = \frac{x}{\sqrt{1+x^2}}$

$f(f(x)) = \frac{f(x)}{\sqrt{1+(f(x))^2}} = \frac{x}{\sqrt{1+2x^2}}$

$f(f(f(x))) = \frac{x}{\sqrt{1+3x^2}}$

So, $\underbrace{f \circ f \circ f \dots \circ f}_{n \text{ times}}(x) = \frac{x}{\sqrt{1+nx^2}}$

Example 124 If $f(x) = x^2 - 2x$ then find the Number of distinct

real c satisfying $f(f(f(f(c)))) = 3$

Solution Given $f(x) = x^2 - 2x$

let $f(f(f(c))) = \theta$

$f(\theta) = 3$

$\theta^2 - 2\theta = 3 \Rightarrow \theta = 3, -1$

let $f(f(c)) = \alpha$

$f(\alpha) = 3, -1$

$\alpha^2 - 2\alpha = 3, \alpha^2 - 2\alpha = -1$

$\alpha = 3, -1 \quad \alpha = 1$

So, $\alpha = 3, -1, 1$

$f(f(c)) = 3, -1, 1$

let $f(c) = \beta$

$f(\beta) = 3, -1, 1 \Rightarrow \beta^2 - 2\beta = 3, -1, 1$

$\beta = 3, -1, 1, 1 + \sqrt{2}, 1 - \sqrt{2}$

$f(c) = 3, -1, 1, 1 + \sqrt{2}, 1 - \sqrt{2}$

↓ ↓ ↓ ↓ ↓

Number of 2 1 2 2 2

solution

Total number of solution 9

Example 125 $f(x) = \frac{1}{(1-x^{2011})^{\frac{1}{2011}}}$, $f(f(f\dots f(x))) = \{-x\}$

f is written 2013 times. Then find the no. of solutions of x .

Solution $f(x) = \frac{1}{\sqrt[2011]{1-x^{2011}}}$

$f \circ f(x) = \left(1 - \frac{1}{x^{2011}}\right)^{\frac{1}{2011}}; x \neq 0$

$= \frac{(1-x^{2011})^{\frac{1}{2011}}}{-x}$

$f \circ f \circ f(x) = x$

It repeats in a cycle of 3

So, $f \circ f \circ f \dots 2013 \text{ times} = x$

$\Rightarrow x = \{-x\}$

$\Rightarrow x = 0 \text{ \& } x = \frac{1}{2}$ (where $x \neq 0$ because of domain)

Required answer $x = 1/2$.

11. HOMOGENEOUS FUNCTIONS

A function is said to be homogeneous with respect to any set of variable when each of its terms is of the same degree with respect to those variables.

For examples $5x^2 + 3y^2 - xy$ is homogenous in x & y . Symbolically if, $f(tx, ty) = t^n f(x, y)$ then $f(x, y)$ is homogeneous function of degree n .

Example 126 $f(x, y) = 4x^2 + y^2$ check it is homogeneous or not ?

Solution $f(x, y) = 4x^2 + y^2$

Put $x = tx$ and $y = ty$

$f(tx, ty) = 4(tx)^2 + (ty)^2$

$f(tx, ty) = 4t^2x^2 + t^2y^2$

$f(tx, ty) = t^2(4x^2 + y^2)$

$\therefore 4x^2 + y^2$ is $f(x, y)$
 $f(tx, ty) = t^2 f(x, y)$

Hence, $f(x, y)$ is homogeneous function with degree 2.

Example 127 $f(x, y) = x^3 + y^2$, check it is homogeneous or not ?

Solution $f(x, y) = x^3 + y^2$

Put $x = tx$ and $y = ty$

$f(tx, ty) = (tx)^3 + (ty)^2$

$f(tx, ty) = t^3 x^3 + t^2 y^2$

$f(tx, ty) = t^2 (tx^3 + y^2)$

But $tx^3 + y^2$ is NOT $f(x, y)$.

So $f(x, y)$ is NOT homogeneous.

Example 128 Which of the following function is not homogeneous ?

(A) $x^3 + 8x^2y + 7y^3$ (B) $y^2 + x^2 + 5xy$

(C) $\frac{xy}{x^2 + y^2}$ (D) $\frac{2x - y + 1}{2y - x + 1}$

Solution Here in above option we check the degree of each term in the equation.

Option (A), (B) & (C) has same degree in each term.

But in option (D) the degree of each term is not same.

So, option (D) is correct.

Example 129 Prove that the following function is homogeneous function of degree 6 : $x^2y^4 + x^3y^3 - \frac{x^7}{y}$

neous function of degree 6 : $x^2y^4 + x^3y^3 - \frac{x^7}{y}$

Solution Let $f(x, y) = x^2y^4 + x^3y^3 - \frac{x^7}{y}$

Put $x = tx$ and $y = ty$

$f(tx, ty) = (tx)^2 (ty)^4 + (tx)^3 (ty)^3 - \frac{(tx)^7}{ty}$

$= t^2 x^2 t^4 y^4 + t^3 x^3 t^3 y^3 - \frac{t^7 x^7}{ty}$

$= t^6 x^2 y^4 + t^6 x^3 y^3 - \frac{t^6 x^7}{y}$

$= t^6 (x^2 y^4 + x^3 y^3 - \frac{x^7}{y})$

Therefore, $f(tx, ty) = t^6 f(x, y)$

BOUNDED FUNCTION

A function is said to be bounded if $|f(x)| \leq M$, where M is a finite quantity.

These are the following examples of the bounded function

Function	Range	Bounded
$\sin x$	$[-1, 1]$	Yes

$\cos x$	$[-1, 1]$	Yes
$\tan x$	\mathbb{R}	No
$\operatorname{cosec} x$	$\mathbb{R} - (-1, 1)$	No
$\sec x$	$\mathbb{R} - (-1, 1)$	No
$\cot x$	\mathbb{R}	No
e^x	$(0, \infty)$	No
$\log x$	\mathbb{R}	No
$[x]$	\mathbb{I}	No
$\{x\}$	$[0, 1)$	Yes
$\operatorname{sgn}(x)$	$\{-1, 0, 1\}$	Yes

12. IMPLICIT & EXPLICIT FUNCTION

A. Implicit function : A function $y = f(x)$ is said to be an implicit function of x if y cannot be written in terms of x only. For example

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

(ii) $xy = \sin(x + y)$.

B. Explicit Function : A function $y = f(x)$ is said to be an explicit function of x if the dependent variable y can be expressed in terms of the independent variable x only. For example,

(i) $y = x - \cos x$ (ii) $y = x + \log_e x - 2x^3$.

Illustration 130 Which of the following function is implicit function ?

(A) $y = x^2$ (B) $xy - \sin(x + y) = 0$

(C) $y = \frac{x^2 \log x}{\sin x}$

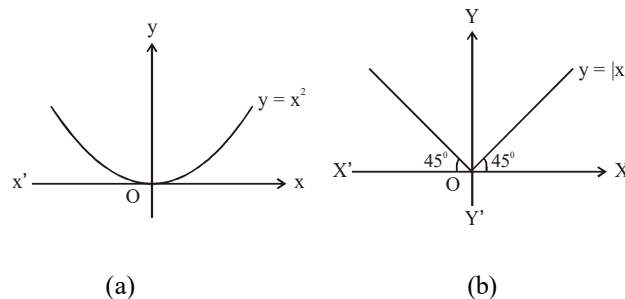
Solution It is clear that in (B) y is not clearly expressed in x .

13. ODDANDEVEN FUNCTION

A. Even Function

A function $y = f(x)$ is said to be an even function if $f(-x) = f(x) \forall x \in D_f$

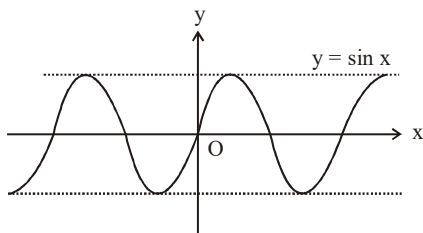
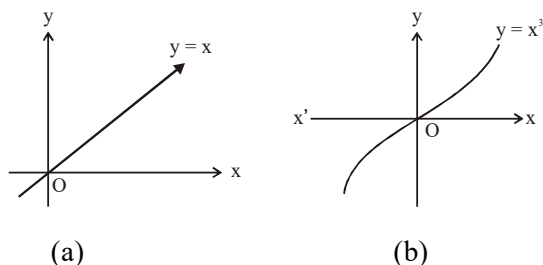
The graph of an even function $y = f(x)$ is symmetrical about the y -axis, i.e., if point (x, y) lies on the graph, then $(-x, y)$ also lies on the graph.



B. Odd Function

A function $y = f(x)$ is said to be an odd function if $f(-x) = -f(x) \forall x \in D_f$

The graph of an odd function $y = f(x)$ is symmetrical in opposite quadrants, i.e., if point (x, y) lies on the graph, then $(-x, -y)$ also lies on the graph.



Properties of Odd and Even Function

- Sometimes, it is easy to prove that $f(x) - f(-x) = 0$ for even functions and $f(x) + f(-x) = 0$ for odd functions.
- A function can be either even or odd or neither.
- Any function (not necessarily even or odd) can be expressed as a sum of even and an odd function, i.e.,

$$f(x) = \left(\frac{f(x) + f(-x)}{2} \right) + \left(\frac{f(x) - f(-x)}{2} \right)$$

$$\text{Let } h(x) = \left(\frac{f(x) + f(-x)}{2} \right) \text{ and } g(x) = \left(\frac{f(x) - f(-x)}{2} \right)$$

It can now be easily shown that $h(x)$ is even and $g(x)$ is odd.

- The first derivative of an even function is an odd function and vice versa.
- $f(x) = 0$ is the only function which is defined on the entire number line and is even and odd at the same time.
- Every even function $y = f(x)$ is many-one for all $x \in D_f$.

$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x) - g(x)$	$f(x)g(x)$	$f(x)/g(x)$	$fg(x)$
Even	Even	Even	Even	Even	Even	Even
Even	Odd	Neither even nor Odd	Neither even nor Odd	Odd	Odd	Even
Odd	Even	Neither even nor Odd	Neither even nor Odd	Odd	Odd	Even
Odd	Odd	Odd	Odd	Even	Even	Odd

Example 131 Which of the following functions is (are) even, odd or neither ?

- $f(x) = x^2 \sin x$
- $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$
- $f(x) = \log \left(\frac{1-x}{1+x} \right)$

Solution

a. $f(-x) = (-x)^2 \sin(-x) = -x^2 \sin x = -f(x)$
Hence, $f(x)$ is odd.

b. $f(-x) = \sqrt{1+(-x)+(-x)^2} - \sqrt{1-(-x)+(-x)^2}$
 $= \sqrt{1-x+x^2} - \sqrt{1+x+x^2}$
 $= -f(x)$
Hence, $f(x)$ is odd.

c. $f(-x) = \log \left\{ \frac{1-(-x)}{1+(-x)} \right\} = \log \left(\frac{1+x}{1-x} \right)$
 $= -f(x)$
Hence, $f(x)$ is odd

Example 132 If $f(x) = (h_1(x) - h_1(-x))(h_2(x) - h_2(-x)) \dots (h_{2n+1}(x) - h_{2n+1}(-x))$ and $f(200) = 0$, then prove that $f(x)$ is a many-one function.

Solution : $f(x) = (h_1(x) - h_1(-x))(h_2(x) - h_2(-x)) \dots (h_{2n+1}(x) - h_{2n+1}(-x))$
 $\therefore f(-x) = (-1)^{2n+1} f(x) = -f(x)$
 or $f(x) + f(-x) = 0$
 So, $f(x)$ is odd. Therefore,
 $f(-200) = -f(200) = 0$
 So, $f(x)$ is many-one.

Example 133 Find whether the given function is even or odd :

$f(x) = \frac{x(\sin x + \tan x)}{\left[\frac{x+\pi}{\pi} \right] - \frac{1}{2}}$; where $[]$ denotes the greatest integer function.

Solution $f(x) = \frac{x(\sin x + \tan x)}{\left[\frac{x+\pi}{\pi} \right] - \frac{1}{2}} = \frac{x(\sin x + \tan x)}{\left[\frac{x}{\pi} \right] + 1 - \frac{1}{2}}$
 $= \frac{x(\sin x + \tan x)}{\left[\frac{x}{\pi} \right] + 0.5}$

$$\Rightarrow f(-x) = \frac{-x\{\sin(-x) + \tan(-x)\}}{\left[-\frac{x}{\pi}\right] + 0.5}$$

$$\therefore = \begin{cases} \frac{x(\sin x + \tan x)}{-1 - \left[\frac{x}{\pi}\right] + 0.5}, & x \neq n\pi \\ 0, & x = n\pi \end{cases}$$

$$\text{Hence, } f(-x) = -\left(\frac{x(\sin x + \tan x)}{\left[\frac{x}{\pi}\right] + 0.5}\right) \text{ and } f(-x) = 0$$

$$f(-x) = -f(x)$$

Hence, $f(x)$ is an odd function if $x \neq n\pi$ and $f(x) = 0$ if $x = n\pi$ is both even and odd function.

Example 134 Determine the nature of the following functions for even and odd :

(i) $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$ (ii) $f(x) = \sin x + \cos x$

(iii) $f(x) = x^2 - |x|$

Solution

(i) We have, $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$

$$\therefore f(-x) = -x\left(\frac{a^{-x} - 1}{a^{-x} + 1}\right)$$

$$= -x\left(\frac{1 - a^x}{1 + a^x}\right)$$

$$= x\left(\frac{a^x - 1}{a^x + 1}\right) = f(x)$$

So, $f(x)$ is an even function.

(ii) We have,

$$f(x) = \sin x + \cos x$$

$$\Rightarrow f(-x) = \sin(-x) + \cos(-x)$$

$$= -\sin x + \cos x$$

Clearly, $-\sin x + \cos x$ is neither equal to $f(x)$ nor equal to $-f(x)$.

So, $f(x)$ is neither even nor odd.

(iii) We have,

$$f(x) = x^2 - |x|$$

$$\Rightarrow f(-x) = (-x)^2 - |-x| = x^2 - |x| = f(x)$$

So, $f(x)$ is an even function.

Example 135 Let f, g be two functions. Then prove that

- (i) f is even, g is even $\Rightarrow fog$ is an even function
- (ii) f is odd, g is odd $\Rightarrow fog$ is an odd function
- (iii) f is even, g is odd $\Rightarrow fog$ is an even function
- (iv) f is odd, g is even $\Rightarrow fog$ is an even function

Solution

(i) We have,

$$(fog)(-x) = f(g(-x))$$

$$= f(g(x)) \quad [\because g \text{ is even } \therefore g(-x) = g(x)]$$

$$= fog(x) \quad \text{for all } x$$

$\therefore fog$ is an even function

(ii) We have,

$$(fog)(-x) = f(g(-x))$$

$$= f(-g(x)) \quad [\because g \text{ is odd } \Rightarrow g(-x) = -g(x)]$$

$$= -f(g(x)) \quad [\because f \text{ is odd}]$$

$$= -fog(x) \quad \text{for all } x$$

$\therefore fog$ is an odd function.

(iii) We have,

$$fog(-x) = f(g(-x))$$

$$= f(-g(x)) \quad [\because g \text{ is odd } \therefore g(-x) = -g(x)]$$

$$= f(g(x)) \quad [\because f \text{ is even } \therefore f(-g(x)) = f(g(x))]$$

$$= fog(x) \quad \text{for all } x.$$

$\therefore fog$ is an even function.

(iv) We have,

$$fog(-x) = f(g(-x))$$

$$= f(g(x)) \quad [\because g \text{ is even } \therefore g(-x) = g(x)]$$

$$= fog(x)$$

$\therefore fog$ is an even function.

Example 136 Find out whether the following function is even or odd.

$$f(x) = \begin{cases} x|x| & , \quad x \leq -1 \\ [1+x] + [1-x] & , \quad -1 < x < 1 \\ -x|x| & , \quad x \geq 1 \end{cases}$$

Solution We have,

$$f(x) = \begin{cases} x|x| & , \quad x \leq -1 \\ [1+x] + [1-x] & , \quad -1 < x < 1 \\ -x|x| & , \quad x \geq 1 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} -x^2 & , \quad x \leq -1 \\ 1 + [x] + 1 + [-x] & , \quad -1 < x < 1 \\ -x^2 & , \quad x \geq 1 \end{cases}$$

$$\left[\begin{array}{l} [1+x] = [x] + 1 \\ [1-x] = [-x] + 1 \end{array} \right]$$

$$\Rightarrow f(x) = \begin{cases} -x^2 & , \quad x \leq -1 \\ 2 + [x] + [-x] & , \quad -1 < x < 1 \\ -x^2 & , \quad x \geq 1 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} -x^2, & x \leq -1 \\ 1, & -1 < x < 0 \\ 2, & x = 0 \\ 1, & 0 < x < 1 \\ -x^2, & x \geq 1 \end{cases}$$

$$\left[\begin{array}{l} \because [x]+[-x] = -1, \text{ if } x \notin Z \\ [x]+[-x] = 0, \text{ if } x \in Z \end{array} \right]$$

It is evident from the definition of the function n or its graph that $f(x)$ is an even function.

DPP 10

Total Marks 32

Time 30 Minute

- Question Number 1, 3, 4, 5, 6. **Marking Scheme** : +3 for correct answer 0 in all other cases. [(5.3)=15]
- Question Number 8. **Marking Scheme** : +3 for correct answer -1 in all other cases. [(1.3)=3]
- Question Number 2. **Marking Scheme** : +6 for correct answer 0 in all other cases. [(2.3)=6]
- Question Number 7. **Marking Scheme** : +8 for correct answer 0 in all other cases. [(4.2)=8]

1. Let $f(x) = ax + b$ and $g(x) = cx + d$, $a \neq 0$ $c \neq 0$. Assume $a = 1$, $b = 2$. If $(fog)(x) = (gof)(x)$ for all x , what can you say about c and d ?

2. Which of the following functions is (are) even, odd or neither ?

(i) $f(x) = \log(x + \sqrt{1 + x^2})$ (ii) $f(x) = \sin x - \cos x$

(iii) $f(x) = \frac{e^x + e^{-x}}{2}$

3. Check whether the function

$h(x) = (\sqrt{\sin x} - \sqrt{\tan x})(\sqrt{\sin x} + \sqrt{\tan x})$ is whether odd or even.

4. If f is an even function defined on the interval $[-5, 5]$, then find the real values of x satisfying the equation

$f(x) = f\left(\frac{x+1}{x+2}\right)$.

5. Determine the nature of the function in terms of odd and even.

$f(x) = \sin(\log(x + \sqrt{x^2 + 1}))$.

6. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function given by

$f(x+y) + f(x-y) = 2f(x)f(y)$ for all $x, y \in \mathbb{R}$ and $f(0) \neq 0$. Prove that $f(x)$ is an even function. What can be said about $f(x)$ if $f(0) = 0$?

7. Find whether the following functions are even or odd or

none :

(i) $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$ (ii) $f(x) = [(x+1)^2]^{1/3} + [(x-1)^2]^{1/3}$

(iii) $f(x) = \frac{(1+2^x)^2}{2^x}$ (iv) $f(x) = x \sin^2 x - x^3$

8. Let $f(x) = \frac{x-3}{x+1}$, $x \neq -1$. Then $f^{2010}(2014)$ [where $f^n(x) =$

f of of (x)] is :
n times

- (A) 2010 (B) 4020
(C) 4028 (D) 2014

Result Analysis

- 25 to 32 Marks : **Advanced Level.**
- 17 to 25 Marks : **Mains Level.**
- < 17 Marks : **Please go through above artical again.**

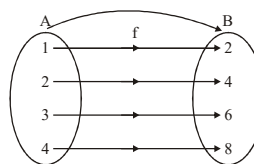
14. INVERSE OF A FUNCTION :

Let $f: A \rightarrow B$ be a one-one & onto function, then there exists a unique function $g: B \rightarrow A$ such that

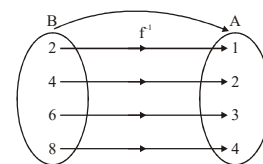
$f(x) = y \Leftrightarrow g(y) = x, \forall x \in A \& y \in B$.

Then g is said to be inverse of f .

Thus $g = f^{-1}: B \rightarrow A = \{(f(x), x) | (x, f(x)) \in f\}$.



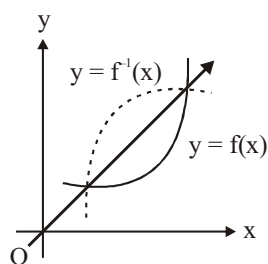
(a)



(b)

Properties of Inverse Functions :

- The inverse of bijective function is unique and bijective.
- Let $f: A \rightarrow B$ be a function such that f is bijective and $g: B \rightarrow A$ is inverse of f . Then $f \circ g = I_B =$ identity function of set B . Then $g \circ f = I_A =$ identity function of set A .
- If $f \circ g = g \circ f$, then either $f^{-1} = g$ or $g^{-1} = f$ and $f \circ g(x) = g \circ f(x) = x$.
- If f and g are two bijective functions such that $f: A \rightarrow B$ and $g: B \rightarrow C$, then $g \circ f: A \rightarrow C$ is bijective. Also, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
- Graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetrical about $y = x$ and to solve equation like $f(x) = f^{-1}(x)$, you may solve $f(x) = x$ instead of solving $f(x) = f^{-1}(x)$.



6. sometimes the solution of $f(x) = f^{-1}(x)$ may not lie on line $y = x$

Example 137 Which of the following functions has inverse function ?

a. $f: Z \rightarrow Z$ defined by $f(x) = x + 2$

b. $f: Z \rightarrow Z$ defined by $f(x) = 2x$

Solution Functions in option (a) is one-one and has range Z , i.e., onto. Hence, it is invertible.

$f: Z \rightarrow Z$ defined by $f(x) = 2x$ is one-one but has only even integers in the range.

Hence, it is not onto.

Example 138 Let $A = \mathbb{R} - \{3\}$, $B = \mathbb{R} - \{1\}$, and let $f: A \rightarrow B$ be

defined by $f(x) = \frac{x-2}{x-3}$. Is invertible? Explain

Solution Let $x_1, x_2 \in A$ and let $f(x_1) = f(x_2)$

$$\text{or } \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$$\text{or } x_1 x_2 - 3x_1 - 2x_2 + 6 = x_1 x_2 - 3x_2 - 2x_1 + 6$$

$$\text{or } x_1 = x_2$$

So, f is one-one.

To find whether f is onto or not, first let us find the range of f .

$$\text{Let } y = f(x) = \frac{x-2}{x-3}$$

$$\text{or } xy - 3y = x - 2$$

$$\text{or } x(y-1) = 3y-2$$

$$\text{or } x = \frac{3y-2}{y-1}$$

x is defined if $y \neq 1$, i.e., the range of f is $\mathbb{R} - \{1\}$ which is also the co-domain of f .

So, function is invertible

Example 139 If $f(x) = (ax^2 + b)^3$, then find the function g such that $f(g(x)) = g(f(x))$.

Solution : $f(g(x)) = g(f(x))$

$$f(x) = (ax^2 + b)^3$$

If $g(x) = f^{-1}(x)$, then

$$y = (ax^2 + b)^3$$

$$\text{or } \sqrt[3]{\frac{y- b}{a}} = x$$

$$\text{or } g(x) = \sqrt[3]{\frac{x^{1/3} - b}{a}}$$

Example 140 Find the inverse of $f(x) = \begin{cases} x & , x < 1 \\ x^2 & , 1 \leq x \leq 4 \\ 8\sqrt{x} & , x > 4 \end{cases}$

Solution Given $f(x) = \begin{cases} x & , x < 1 \\ x^2 & , 1 \leq x \leq 4 \\ 8\sqrt{x} & , x > 4 \end{cases}$

Let $f(x) = y$

or $x = f^{-1}(y)$ (i)

$$\therefore \begin{cases} y & , y < 1 \\ \sqrt{y} & , 1 \leq \sqrt{y} \leq 4 \\ y^2 / 64 & , y^2 / 64 > 4 \end{cases} = \begin{cases} y & , y < 1 \\ \sqrt{y} & , 1 \leq y \leq 16 \\ y^2 / 64 & , y > 16 \end{cases}$$

$$f^{-1}(y) = \begin{cases} y & , y < 1 \\ \sqrt{y} & , 1 \leq y \leq 16 \\ y^2 / 64 & , y > 16 \end{cases}$$

$$\text{Hence, } f^{-1}(x) = \begin{cases} x & , x < 1 \\ \sqrt{x} & , 1 \leq x \leq 16 \\ x^2 / 64 & , x > 16 \end{cases}$$

Example 141 Solve the equation $x^2 - x + 1 = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$,

where $x \geq \frac{3}{4}$.

Solution $f(x) = x^2 - x + 1$ and $g(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$ are inverse of

one another.

So, $f(x) = g(x)$.

When $f(x) = x$,

$$x^2 - x + 1 = x$$

$$x^2 - 2x + 1 = 0$$

$$(x-1)^2 = 0$$

$$x = 1$$

Example 142 Show that the function $f: \mathbb{R} - \{0\} \rightarrow \mathbb{R} - \{0\}$

given by $f(x) = \frac{k}{x}$, where k is a non-zero real number, is inverse of itself.

Solution Clearly, $f: \mathbb{R} - \{0\} \rightarrow \mathbb{R} - \{0\}$ given by $f(x) = \frac{k}{x}$ is a

bijection and hence it is invertible.

Let $f(x)=y$

Then, $f(x)=y$

$$\Rightarrow \frac{k}{x} = y \quad \Rightarrow x = \frac{k}{y}$$

$$\Rightarrow f^{-1}(y) = \frac{k}{y} \quad \therefore f^{-1}(x) = \frac{k}{x}$$

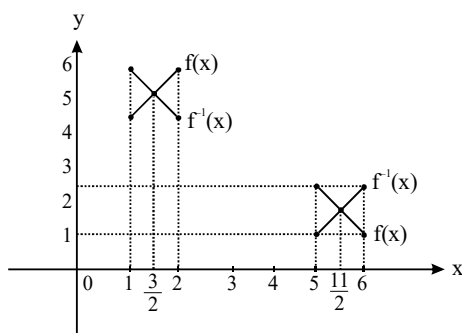
Hence, $f(x) = f^{-1}(x)$ i.e., $f(x)$ is inverse of itself.

Example 143 Let $f(x) = \begin{cases} x+4 & 1 \leq x \leq 2 \\ 7-x & 5 \leq x \leq 6 \end{cases}$

Find number of solution of $f(x) = f^{-1}(x)$

Solution $f^{-1}(x) = \begin{cases} x-4 & 5 \leq x \leq 6 \\ 7-x & 1 \leq x \leq 2 \end{cases}$

Draw graph of $f(x)$ and $f^{-1}(x)$



Clearly both the solutions $x = 3/2$ and $x = 11/2$ is not lying on line $y = x$

Reason : If (a, b) lies on $f(x)$ where $a \neq b$ then (b, a) also lies on $f(x)$ then all solutions of $f(x) = f^{-1}(x)$ will not lie on $y = x$.

Example 144 $f(x) = 1 - x^3$, Find number of solution of $f(x) = f^{-1}(x)$

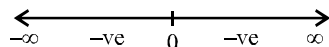
Solution $f(x) = 1 - x^3$

$$\Rightarrow y = 1 - x^3 \quad \Rightarrow x^3 = 1 - y \quad \Rightarrow x = (1 - y)^{1/3}$$

$$\Rightarrow f^{-1}(x) = (1 - x)^{1/3}$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow -\infty} f^{-1}(x) = \infty$$

Now, $f'(x) = -3x^2$



Sign scheme of $f'(x)$

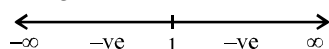
At $x = 0$ we get point of inflection

$$\therefore f''(x) = -6x$$

on $x \in (-\infty, 0) \Rightarrow f''(x) > 0 \Rightarrow$ concave up

on $x \in (0, \infty) \Rightarrow f''(x) < 0 \Rightarrow$ concave down

$$\Rightarrow (f^{-1}(x))' = -\frac{1}{3}(1-x)^{-2/3}$$



Sign scheme of $(f^{-1}(x))'$

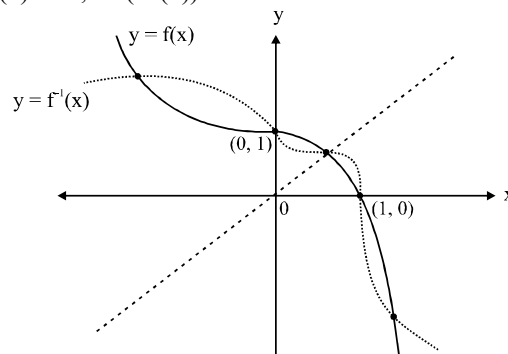
$$\therefore (f^{-1}(x))'' = -\frac{2}{9(1-x)^{5/3}}$$

on $x \in (-\infty, 1) \Rightarrow f''(x) < 0 \Rightarrow$ concave down

on $x \in (1, \infty) \Rightarrow f''(x) > 0 \Rightarrow$ concave up

$$f'(0) = 0, \quad (f^{-1}(0))' = -\frac{1}{3}$$

$$f'(1) = -3, \quad (f^{-1}(1))' = 0$$



Total solutions = 5

15. PERIODIC FUNCTION

Periodic Functions

A function $f(x)$ is said to be a periodic function if there exists a positive real number T such that

$$f(x + T) = f(x) \text{ for all } x \in \mathbb{R}.$$

We know that

$$\sin(x + 2\pi) = \sin(x + 4\pi) = \dots = \sin x$$

and, $\cos(x + 2\pi) = \cos(x + 4\pi) = \dots = \cos x$ for all $x \in \mathbb{R}$.

Therefore, $\sin x$ and $\cos x$ are periodic functions.

Period of Function

If $f(x)$ is periodic function, then the smallest positive real number T is called the period or fundamental period of function $f(x)$ if

$$f(x + T) = f(x) \text{ for all } x \in \mathbb{R}.$$

In order to check the periodicity of a function $f(x)$, we follow the following algorithm.

Important facts about periodic function

1. If $f(x)$ is a periodic function with periodic T and $a, b \in \mathbb{R}$ such that $a \neq 0$, then $af(x) + b$ is periodic with period T .
2. If $f(x)$ is a periodic function with period T and $a, b \in \mathbb{R}$ such that $a \neq 0$, then $f(ax + b)$ is periodic with period $T/|a|$.
3. Let $f(x)$ and $g(x)$ be two periodic functions such that :

Period of $f(x) = \frac{m}{n}$, where $m, n \in \mathbb{N}$ and m, n are co-prime.

and,

Period of $g(x) = \frac{r}{s}$, where $r \in \mathbb{N}$ and $s \in \mathbb{N}$ are co-prime

Then, $(f + g)(x)$ is periodic with period T given by

$$T = \frac{\text{LCM of } (m, r)}{\text{HCF of } (m, s)}$$

Provided that there does not exist a positive number $k < T$ for which $f(k+x) = g(x)$ and $g(k+x) = f(x)$, else k will be the period of $(f+g)(x)$.

The above result is also true for functions $\frac{f}{g}$, $f-g$ and fg .

4. A constant function is periodic but does not have a well-defined period.
5. If g is periodic, then fog will always be a periodic function. The period of fog may not be period of g .
6. A continuous periodic function is bounded.
7. If $f(x)$ and $g(x)$ are periodic functions with periods T_1 and T_2 , respectively, then $h(x) = f(x) + g(x)$ has period as
 - (a) LCM of $\{T_1, T_2\}$; if $f(x)$ and $g(x)$ cannot be interchanged by adding a least positive number less than the LCM of $\{T_1, T_2\}$.
 - (b) k ; if $f(x)$ and $g(x)$ can be interchanged by adding a least positive number k ($k < \text{LCM of } \{T_1, T_2\}$).

For example, consider the function $f(x) = |\sin x| + |\cos x|$. Now, $|\sin x|$ and $|\cos x|$ have period π . Hence, according to the rule of LCM, the period of $f(x)$ is π . but

$$\begin{aligned} f\left(x + \frac{\pi}{2}\right) &= \left| \sin\left(x + \frac{\pi}{2}\right) \right| + \left| \cos\left(x + \frac{\pi}{2}\right) \right| \\ &= |\cos x| + |\sin x| \end{aligned}$$

Hence, the period of $f(x)$ is $\pi/2$.

Some Important Points To Remember

- (i) $\sin^n x$, $\cos^n x$, $\sec^n x$, $\text{cosec}^n x$ are periodic functions with period 2π and π according as n is odd or even.
- (ii) $\tan^n x$, $\cot^n x$ are periodic functions with period π whether n is even or odd.
- (iii) $|\sin x|$, $|\cos x|$, $|\tan x|$, $|\cot x|$, $|\sec x|$, $|\text{cosec } x|$ are periodic with period π .
- (iv) $|\sin x| + |\cos x|$, $|\tan x| + |\cot x|$, $|\sec x| + |\text{cosec } x|$ are periodic with period $\pi/2$.
- (v) $\sin^{-1}(\sin x)$, $\cos^{-1}(\cos x)$, $\text{cosec}^{-1}(\text{cosec } x)$, $\sec^{-1}(\sec x)$ are periodic with period 2π whereas $\tan^{-1}(\tan x)$ and $\cot^{-1}(\cot x)$ are periodic with period π .

Example 145 Find the periods (if periodic) of the following functions ($[.]$ denotes the greatest integer function).

- a. $f(x) = e^{\log(\sin x)} + \tan^3 x - \text{cosec}(3x - 5)$
- b. $f(x) = x - [x - b]$, $b \in \mathbb{R}$
- c. $f(x) = \tan \frac{\pi}{2} [x]$

Solution

- a. $f(x) = e^{\log(\sin x)} + \tan^3 x - \text{cosec}(3x - 5)$
The period of $e^{\log(\sin x)}$ is 2π , of $\tan^3 x$ is π , and of $\text{cosec}(3x - 5)$ is $\frac{2\pi}{3}$. So,

$$\text{Period} = \text{LCM of } \left\{ 2\pi, \pi, \frac{2\pi}{3} \right\} = 2\pi$$

- b. $f(x) = x - [x - b] = b + \{x - b\}$ (\because Period of $\{.\}$ is 1)
So, $f(x)$ has period 1.
- c. $f(x) = \tan \frac{\pi}{2} [x]$

$$\text{or } \tan \frac{\pi}{2} [x + T] = \tan \frac{\pi}{2} [x]$$

$$\text{or } \frac{\pi}{2} [x + T] = n\pi + \frac{\pi}{2} [x]$$

$$\text{or } \text{Period} = 2 \text{ (Least positive value)}$$

Example 146 If $f(x) = \sin x + \cos ax$ is a periodic function, show that a is a rational number.

Solution : Period of $\sin x = 2\pi = \frac{2\pi}{1}$ and period of $\cos ax = \frac{2\pi}{|a|}$

$$\therefore \text{Period of } \sin x + \cos ax = \text{LCM of } \frac{2\pi}{1} \text{ and } \frac{2\pi}{|a|}$$

$$= \frac{\text{LCM of } 2\pi \text{ and } 2\pi}{\text{HCF of } 1 \text{ and } a} = \frac{2\pi}{\lambda}$$

where λ is the HCF of 1 and a .

Since λ is the HCF of 1 and a , $\frac{1}{\lambda}$ and $\frac{|a|}{\lambda}$ should both be integers.

Suppose $\frac{1}{\lambda} = p$ and $\frac{|a|}{\lambda} = q$. Then,

$$\frac{|a|}{\frac{1}{p}} = \frac{q}{p}, \text{ where } p, q \in \mathbb{Z}$$

$$\text{i.e., } |a| = \frac{q}{p}$$

Hence, a is the rational number.

Example 147 For what integral value of n is 3π the period of

the function $\cos(nx) \sin\left(\frac{5x}{n}\right)$?

Solution : Let $f(x) = \cos nx \sin\left(\frac{5x}{n}\right)$. $f(x)$ be periodic. Then,

$$f(x + \lambda) = f(x), \text{ where } \lambda \text{ is period}$$

or $\cos(n\lambda) \sin\left(\frac{5x+5\lambda}{n}\right) = \cos nx \sin\left(\frac{5x}{n}\right)$

At $x=0$,

$\cos n\lambda \sin\left(\frac{5\lambda}{n}\right) = 0$

If $\cos n\lambda = 0$, then

$n\lambda = r\pi + \frac{\pi}{2}, r \in I$

or $n(3\pi) = r\pi + \frac{\pi}{2}$

or $3n - r = \frac{1}{2}$ (Impossible)

Again, let $\sin\left(\frac{5\pi}{n}\right) = 0$.

Then, $\frac{5\lambda}{n} = p\pi \quad (p \in I)$

or $\frac{5(3\pi)}{n} = p\pi$

or $n = \frac{15}{p}$

For $p = \pm 1, \pm 3, \pm 5, \pm 15$, we have, respectively,
 $n = \pm 15, \pm 5, \pm 3, \pm 1$

Example 148 Let f be a real valued function defined for all real numbers x such that for some fixed $a > 0$,

$f(x+a) = \frac{1}{2} + \sqrt{f(x) - \{f(x)\}^2}$ for all x .

Show that the function $f(x)$ is periodic with period $2a$.

Solution We have, $f(x+a) = \frac{1}{2} + \sqrt{f(x) - \{f(x)\}^2}$

$\therefore f(x+a+a) = \frac{1}{2} + \sqrt{f(x+a) - \{f(x+a)\}^2}$

$\Rightarrow f(x+2a) = \frac{1}{2} + \sqrt{f(x+a)\{1-f(x+a)\}}$

$f(x+2a) = \frac{1}{2} + \sqrt{\left\{\frac{1}{2} + \sqrt{f(x) - \{f(x)\}^2}\right\} \left\{\frac{1}{2} - \sqrt{f(x) - \{f(x)\}^2}\right\}}$

$\Rightarrow f(x+2a) = \frac{1}{2} + \sqrt{\frac{1}{4} - \{f(x) - \{f(x)\}^2\}}$

$\Rightarrow f(x+2a) = \frac{1}{2} + \sqrt{\frac{1}{4} - f(x) + \{f(x)\}^2}$

$\Rightarrow f(x+2a) = \frac{1}{2} + \sqrt{\left\{f(x) - \frac{1}{2}\right\}^2}$

$\Rightarrow f(x+2a) = \frac{1}{2} + \left|f(x) - \frac{1}{2}\right|$

$\Rightarrow f(x+2a) = \frac{1}{2} + f(x) - \frac{1}{2}$

$\Rightarrow f(x+2a) = f(x)$ for all x .

Hence, $f(x)$ is a periodic function with period $2a$.

Example 149 Find the period of function

$f(x) = \sin x + \tan \frac{x}{2} + \sin \frac{x}{2^2} + \tan \frac{x}{2^3} + \dots + \sin \frac{x}{2^{n-1}} + \tan \frac{x}{2^n}$.

Solution : We have,

$f(x) = \sum_{r=1}^n \left(\sin \frac{x}{2^{r-1}} + \tan \frac{x}{2^r} \right)$
 $= \sum_{r=1}^n f_r(x)$, where $f_r(x) = \sin \frac{x}{2^{r-1}} + \tan \frac{x}{2^r}$

Since each of \sin and \tan

with period $2^r \pi$.

But, \dots . Therefore, $f(x)$ is periodic with period

\dots

\dots

\dots

$T = 2^n \pi$

Example 150 Find the period of $f(x) = 5 \sin 3x - 7 \sin 8x$.

Solution : We observe that :

Period of $5 \sin 3x$ is \dots

Period of $7 \sin 8x$ is \dots

\therefore Period of $f(x) = \dots$

Example 151 Find the period of the function

$f(x) = e^{x-[x]} + |\cos \pi x| + |\cos 2\pi x| + \dots + |\cos n\pi x|$

Solution : We observe that

Period of $x - [x]$ is 1

Period of $|\cos \pi x|$ is

Period of $|\cos 2\pi x|$ is

Period of $|\cos 3\pi x|$ is

Finally, Period of $|\cos n\pi x| =$

\therefore Period of $f(x) = \text{LCM of}$

Period of $f(x) =$

Period of

Example 152 Let $f(x)$ be periodic and k be a positive real number such that $f(x+k) + f(x) = 0$ for all $x \in \mathbb{R}$. Prove that $f(x)$ is periodic with period $2k$.

Solution : We have $f(x+k) + f(x) = 0$

or $f(x+k) = -f(x)$

Put $x = x+k$.

Then,

or $f(x+2k) = -f(x+k)$

or $f(x+2k) = f(x)$,

which clearly shows that $f(x)$ is periodic with period $2k$.

Example 153 If $f(x)$ satisfies the relation $f(x) + f(x+4) = f(x+2) + f(x+6)$ for all x , then prove that $f(x)$ is periodic and find its period.

Solution : Given $f(x) + f(x+4) = f(x+2) + f(x+6)$... (i)

Replace x by $x+2$. Then

$f(x+2) + f(x+6) = f(x+4) + f(x+8)$ (ii)

From (i) and (ii), we have $f(x) = f(x+8)$.

So, time period of function is 8.

Example 154 Let $f : \mathbb{R} \rightarrow \mathbb{R} - \{3\}$ be a function with the property

that there exist $T > 0$ such that $f(x+T) =$ for every

$x \in \mathbb{R}$. Prove that $f(x)$ is periodic.

Solution We have, $f(x+T) =$

For period check at multiple of T

i.e., $2T, 4T, 6T, \dots$

Replace x by $x+T$

$f(x+2T) = \alpha]$

$\Rightarrow \Rightarrow$

$\Rightarrow f(x+2T) = \neq f(x)$ (ii)

So, $2T$ is not a period of function

Put $x = x+2T$ in equ. (ii)

$f(x+4T) = \Rightarrow$

$\Rightarrow \Rightarrow$

$\Rightarrow f(x+4T) = \alpha$

Hence, $f(x+4T) = f(x)$

So period is $4T$

TRANSFORMATION OF GRAPHS

1. Vertical shift : $f(x)$ transforms to $f(x) \pm a$, i.e.,

- a. $f(x) \rightarrow f(x) + a$ shifts the given graph of $f(x)$ upward through a units.
- b. $f(x) \rightarrow f(x) - a$, shifts the given graph of $f(x)$ downward through a units.

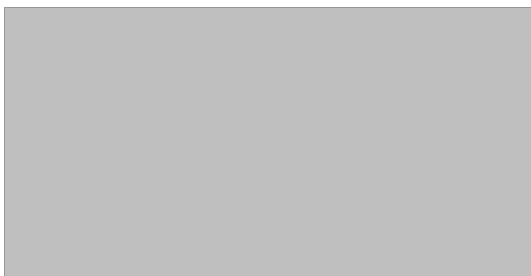
Graphically, it could be stated as shown in the figure :



2. Horizontal shift : $f(x)$ transforms to $f(x \pm a)$, i.e.,

- a. $f(x) \rightarrow f(x-a)$; a is positive, shifts the graph of $f(x)$ through a units towards right.
- b. $f(x) \rightarrow f(x+a)$; a is positive, shifts the graph of $f(x)$ through a units towards left.


Graphically, it could be stated as shown in the figure.



3. **Horizontal stretch** : $f(x)$ transforms to $f(ax)$, i.e.,



a. $f(x) \rightarrow f(ax)$; $a > 1$. shrinks (or contracts) the graph of $f(x)$ a times along the x -axis.

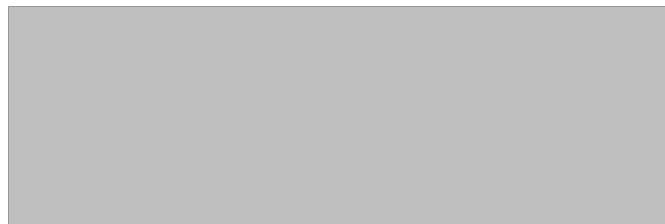
b. $f(x) \rightarrow$ 

$f(x)$ a times along the x -axis.

Graphically, it can be stated as shown in fig.

Example 155 Plot $y = \sin x$ and $y = \sin 2x$.


Solution Here, $y = \sin 2x$, So, shrink (or contract) the graph of $\sin x$ by a factor of 2 along the x -axis.



From fig, $\sin x$ is periodic with period 2π and $\sin 2x$ is periodic with period π .

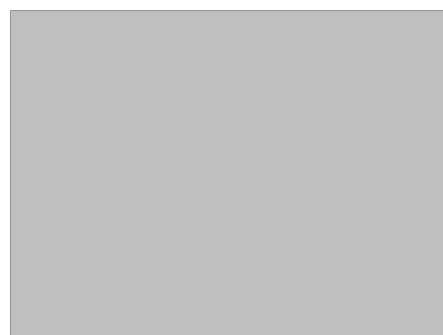
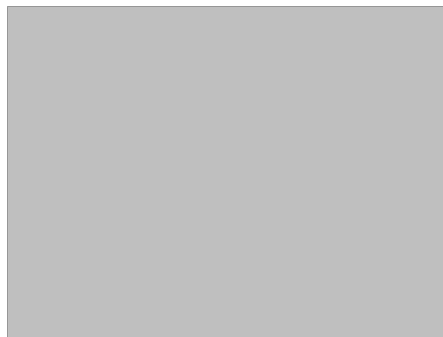
4. **Vertical stretch** : $f(x)$ transforms to $af(x)$. It is clear that the corresponding points (points with the same x coordinates) would have their ordinates in the ratio of 1 : a .



Example 156 Consider the function 

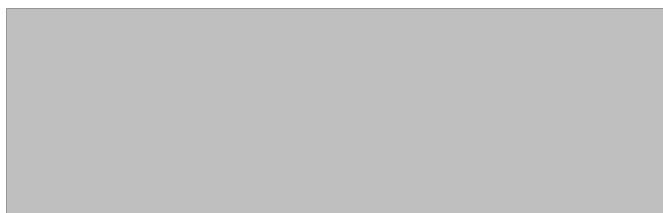
Then draw the graph of the function $y = f(x)$, $y = f(|x|)$, $y = |f(x)|$, and $y = |f(|x|)$.

Solution



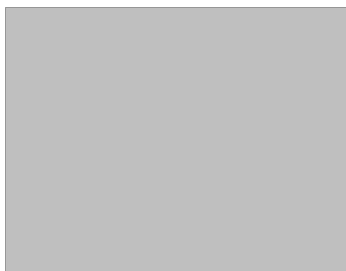
Example 157 Plot $y = \sin x$ and $y = 2 \sin x$.

Solution We know $y = \sin x$ and $f(x) \rightarrow af(x)$.

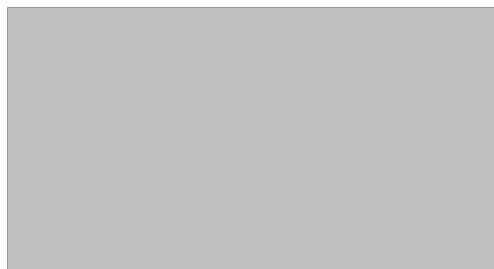


So, stretch the graph of $f(x)$ a times along the y-axis.
 Here, $y = 2 \sin x$.
 So, stretch the graph of $\sin x$, 2 times along the y-axis.

5. **Horizontal flip** : $f(x)$ transforms to $f(-x)$, i.e.,
 $f(x) \rightarrow f(-x)$
 To draw $y = f(-x)$, take the image of the curve $y = f(x)$ in the y-axis as plane mirror.
Or
 Turn the graph of $f(x)$ by 180° about the y-axis.
 Graphically, it is shown as in fig.



Example 158 Plot the curve $y = \log_e(-x)$.
Solution Here, $y = \log_e(-x)$; take the mirror image of $y = \log_e x$ about the y-axis. Graphically, it is shown as in fig.

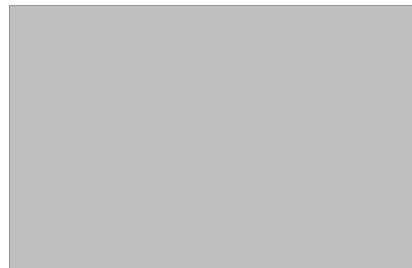


6. **Vertically flip** : $f(x)$ transforms to $-f(x)$ i.e.,
 $f(x) \rightarrow -f(x)$
 To draw $y = -f(x)$, take the image of $y = f(x)$ in the x-axis as plane mirror.
OR
 Turn the graph of $f(x)$ by 180° about the x-axis.

7. **$f(x)$ transforms to $-f(-x)$**
 That is, $f(x) \rightarrow -f(-x)$,
 To draw $y = -f(-x)$, take the image of $f(x)$ about the y-axis to obtain $f(-x)$ and then the image of $f(-x)$ about the x-axis to obtain $-f(-x)$. So, for the transformation
 $f(x) \rightarrow -f(-x)$
 do the following :

- a. Image about the y-axis.

- b. Image about the x-axis.
 Graphically, it is shown as the fig.



You can do all the above transformations in one go using $af(b(x+c))+d$

a is vertical stretch/compression-flip

- $|a| > 1$ stretches
- $|a| < 1$ compresses
- $a < 0$ flip the graph upside down

b is horizontal stretch/compression-flip

- $|b| > 1$ compresses
- $|b| < 1$ stretches
- $b < 0$ flips the graph left-right

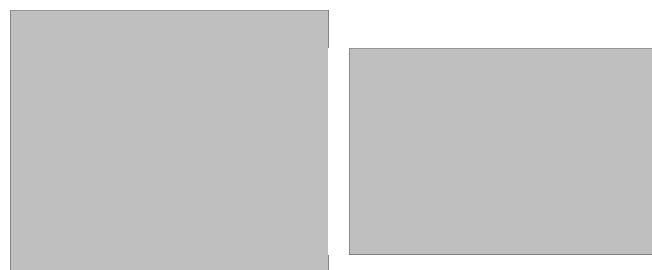
c is horizontal shift

- $c < 0$ shifts to the right
- $c > 0$ shifts to the left

d is vertical shift

- $d > 0$ shifts upwards
- $d < 0$ shifts downward

8. **$f(x)$ transforms to $y = |f(x)|$**
 $|f(x)| = f(x)$ if $f(x) \geq 0$ and $|f(x)| = -f(x)$ if $f(x) < 0$.
 It means that the graph of $f(x)$ and $|f(x)|$ would coincide if $f(x) \geq 0$ the upward direction.
 Figure would make the procedure clear.

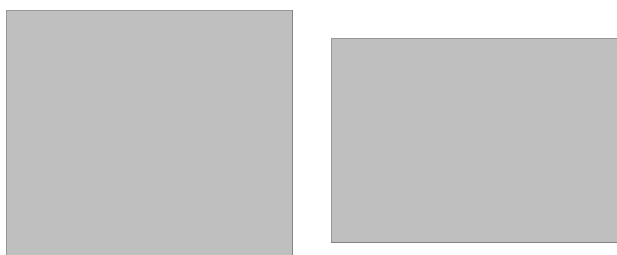


Example 159 Draw the graph for $y = |\log x|$.
Solution To draw the graph for $y = |\log x|$, we have to follow two steps :

- a. Leave the +ve part of $y = \log x$ as it is.
 b. Take the images of the -ve part of $y = \log x$, i.e., the part below the x-axis in the x-axis as plane mirror. Graphically, it is shown as in fig.

Graph of $y = \log x$

Graph of $y = |\log x|$

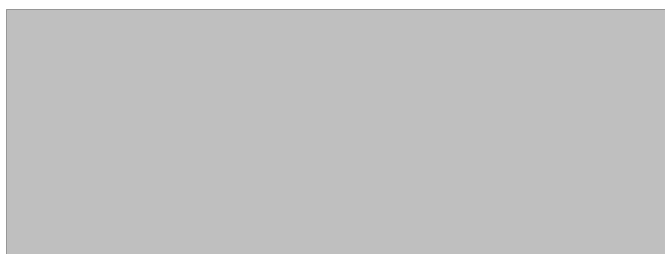


$y = \log_e x$ is differentiable for all $x > 0$ (see fig (a))
 $y = |\log_e x|$ is clearly differentiable for all $x \neq 1$ as at $x = 1$, there is a sharp edge (see fig (b))

Example 160 Sketch the graph for $y = |\sin x|$ and $y = |\cos x|$.

Solution Here, $y = \sin x$ is known.

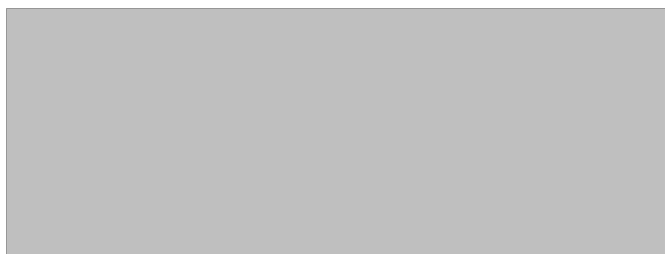
So, to draw $y = |\sin x|$, we take the mirror image (in the x-axis) of the part of the graph of $\sin x$ which lies below the x-axis.



From the above figure, it is clear that $y = |\sin x|$ is differentiable for all $x \neq n\pi$, $n \in \mathbb{Z}$

Here, $y = \cos x$ is known.

So, to draw $y = |\cos x|$, we take the mirror image (in the x-axis) of the part of the graph of $\cos x$ which lies below the x-axis.



9. f(x) transforms to f(|x|)

That is, $f(x) \rightarrow f(|x|)$

If we know $y = f(x)$, then to plot $y = f(|x|)$, we would follow two steps :

- a. Leave the graph lying to the right side of the y-axis as it is.
- b. Take the image of $f(x)$ to the right of the y-axis with the y-axis as the plane mirror and the graph of $f(x)$ lying to the left side of the y-axis (if it exists) is omitted.

OR

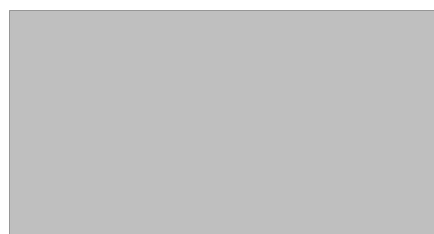
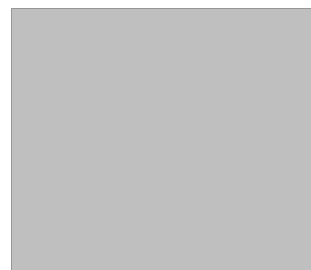
Neglect the curve for $x < 0$ and take the images of curves for $x > 0$



Example 161 Sketch the curve $y = \log |x|$.

Solution As we know the curve $y = \log x$, the curve $y = \log |x|$ could be drawn in two steps :

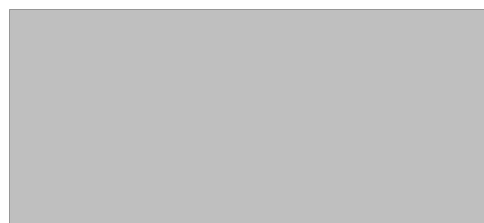
- a. Leave the graph lying to the right side of y-axis as it is.
- b. Take the image of $f(x)$ in the y-axis as plane mirror.



Example 162 Sketch the curve $y = |\log x|$ and $y = |\log |x||$



(a)



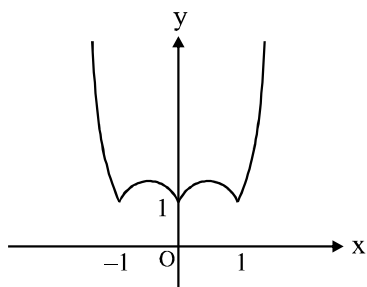
(b)

Example 163 Sketch the curve $f(x) = |x| + |x^2 - 1|$

We know that

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \text{ and } |x^2 - 1| = \begin{cases} (x^2 - 1) & x \in (-\infty, 1] \cup [1, \infty) \\ 1 - x^2 & -1 < x < 1 \end{cases}$$

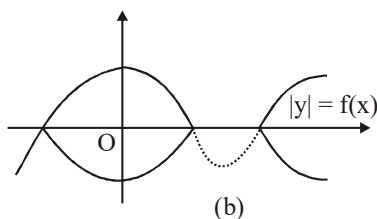
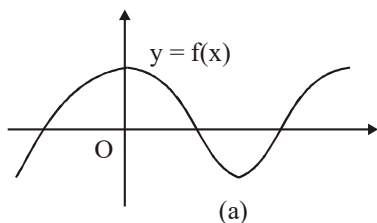
$$f(x) = \begin{cases} x^2 - x - 1 & \text{if } x < -1 \\ -x^2 - x + 1 & \text{if } -1 \leq x < 0 \\ -x^2 + x + 1 & \text{if } 0 \leq x < 1 \\ x^2 + x - 1 & \text{if } x \geq 1 \end{cases}$$



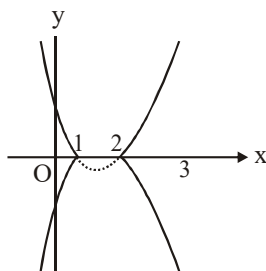
10 Drawing the graph of $|y| = f(x)$ from the known graph of $y = f(x)$

Clearly, $|y| \geq 0$. If $f(x) < 0$, the graph of $|y| = f(x)$ would not exist. Also, if $f(x) \geq 0$, $|y| = f(x)$ would give $y = \pm f(x)$.

Hence, the graph of $|y| = f(x)$ would exist only in the regions where $f(x)$ is non-negative and will be reflected about the x -axis only in those regions. Regions where $f(x) < 0$ will be neglected.



Example 164 Sketch the curve $|y| = (x - 1)(x - 2)$.



11. Drawing the graph of $y = [f(x)]$ from the known graph of $y = f(x)$

MISCELLANEOUS ILLUSTRATION

Illustration 1 Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, 1, 2, 3, 4, 5, 6\}$.

Consider a rule $f(x) = x^2$. Then, Find the domain and range of the function if it exists ?

Solution Consider a rule $f(x) = x^2$. Then,

$$f(-2) = (-2)^2 = 4, \quad f(-1) = (-1)^2 = 1,$$

$$f(0) = 0^2 = 0 \quad f(1) = 1^2 = 1$$

$$\text{and } f(2) = 2^2 = 4$$

Clearly, each element of A is associated to a unique element of B.

So, $f: A \rightarrow B$ given by $f(x) = x^2$ is a function.

Clearly, domain (f) = A = $\{-2, -1, 0, 1, 2\}$ and range (f) = $\{0, 1, 4\}$.

Illustration 2 If $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function? If this is described by the formula, $g(x) = \alpha x + \beta$, then what values should be assigned to α and β ?

Solution Since no two ordered pairs in g have the same first component. So, g is a function such that

$$g(1) = 1, g(2) = 3, g(3) = 5 \text{ and } g(4) = 7$$

It is given that $g(x) = \alpha x + \beta$. Therefore,

$$g(1) = 1 \text{ and } g(2) = 3$$

$$\Rightarrow \alpha + \beta = 1 \text{ and } 2\alpha + \beta = 3$$

$$\Rightarrow \alpha = 2, \beta = -1$$

Illustration 3 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(x) = 2^x$. Determine

(a) Range of f (b) $\{x: f(x) = 1\}$

(c) whether $f(x+y) = f(x) \cdot f(y)$ holds.

Solution

(a) Since 2^x is positive for every $x \in \mathbb{R}$. So, $f(x) = 2^x$ is a positive real number for every $x \in \mathbb{R}$. Moreover, for every positive real number x, there exist $\log_2 x \in \mathbb{R}$ such that

$$f(\log_2 x) = 2^{\log_2 x} = x$$

Hence, we conclude that the range of f is the set of all positive real numbers.

(b) $\because f(x) = 1 \Rightarrow 2^x = 1 \Rightarrow 2^x = 2^0 \Rightarrow x = 0.$

$$\therefore \{x: f(x) = 1\} = \{0\}.$$

(c) $\because f(x+y) = 2^{x+y} = 2^x \cdot 2^y = f(x) \cdot f(y)$

$$\therefore f(x+y) = f(x) \cdot f(y) \text{ holds for all } x, y \in \mathbb{R}$$

Illustration 4 Find the range of $f(x) = \sec\left(\frac{\pi}{4} \cos^2 x\right)$, where $-\infty < x$

Solution $f(x) = \sec\left(\frac{\pi}{4} \cos^2 x\right)$

We know that $0 \leq \cos^2 x \leq 1$ i.e.,

$$0 \leq \frac{\pi}{4} \cos^2 x \leq \frac{\pi}{4}$$

For the above value of $\theta = \frac{\pi}{4} \cos^2 x$, $\sec x$ is an increasing function.

At $\cos x = 0$, $f(x) = 1$ and at $\cos x = 1$, $f(x) = \sqrt{2}$. Therefore,

$$1 \leq x \leq \sqrt{2} \text{ or } x \in [1, \sqrt{2}]$$

Illustration 5 Find the range of $f(x) = \frac{1}{1-3\sqrt{1-\sin^2 x}}$

Solution $f(x) = \frac{1}{1-3\sqrt{1-\sin^2 x}}$

$$= \frac{1}{1-3\sqrt{\cos^2 x}}$$

$$= \frac{1}{1-3|\cos x|}$$

Now, $-3|\cos x| \in [-3, 0]$

or $1-3|\cos x| \in [-2, 1]$

or $\frac{1}{1-3|\cos x|} \in (-\infty, -1/2] \cup [1, \infty)$

Illustration 6 The function $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = \cos^2 x + \sin^4 x$ for $x \in \mathbb{R}$. Then the range of f(x) is

(A) $\left[\frac{3}{4}, 1\right]$ (B) $\left[\frac{3}{4}, 1\right)$

(C) $\left[\frac{3}{4}, 1\right)$ (D) $\left(\frac{3}{4}, 1\right]$

Solution (C)

$$y = f(x) = \cos^2 x + \sin^4 x$$

$$= \cos^2 x + \sin^2 x(1 - \cos^2 x)$$

$$= \cos^2 x + \sin^2 x - \sin^2 x \cos^2 x$$

$$= 1 - \sin^2 x \cos^2 x$$

$$= 1 - \frac{1}{4} \sin^2 2x$$

$$\therefore \frac{3}{4} \leq f(x) \leq 1 \quad (\because 0 \leq \sin^2 2x \leq 1)$$

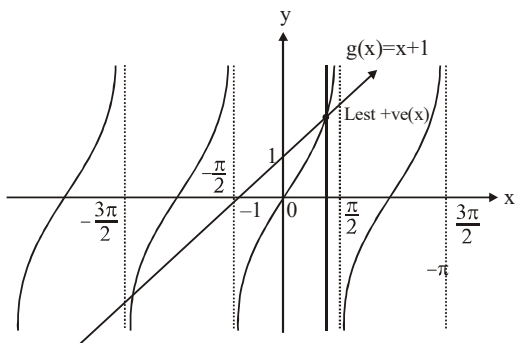
$$\therefore f(x) \in [3/4, 1]$$

Illustration 7 Prove that the least positive value of x, satisfying

$$\tan x = x + 1, \text{ lies in the interval } \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

Solution Let $f(x) = \tan x$ and $g(x) = x + 1$, which can be shown in

follows.



From the figure, $\tan x = x + 1$ has infinitely many solutions

but the least positive value of x lies in $(\frac{\pi}{4}, \frac{\pi}{2})$

Illustration 8 Draw the graph of $y = (\sin 2x)\sqrt{1 + \tan^2 x}$, Find its domain and range.

Solution

$$y = (\sin 2x)\sqrt{1 + \tan^2 x}$$

$$= (2 \sin x \cos x) |\sec x|$$

$$= \begin{cases} 2 \sin x, \cos x > 0 \\ -2 \sin x, \cos x < 0 \end{cases}$$

$$= \begin{cases} 2 \sin x, x \in 1^{\text{st}} \text{ and } 4^{\text{th}} \text{ quadrant} \\ -2 \sin x, x \in 2^{\text{nd}} \text{ and } 3^{\text{rd}} \text{ quadrant} \end{cases}$$

The graph of the function is shown in the following figure.

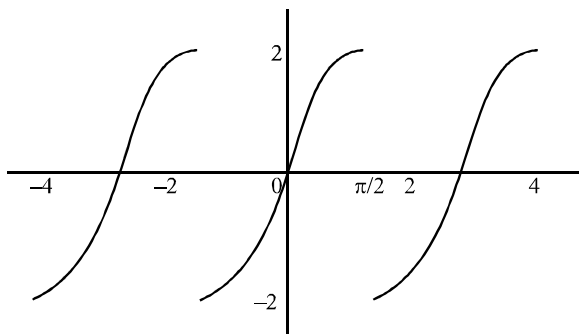


Illustration 9 Find domain of $f(x) = \sqrt{4^x + 8^{(2/3)(x-2)} - 13 - 2^{2(x-1)}}$

Solution

$$4^x + 8^{\frac{2}{3}(x-2)} - 13 - 2^{2(x-1)} \geq 0$$

or $4^x + \frac{4^x}{16} - \frac{4^x}{4} \geq 13$

or $4^x \geq 4^2$ or $x \in [2, \infty)$

Illustration 10 Find domain of $f(x) = \sqrt{\log_{10} \left\{ \frac{\log_{10} x}{2(3 - \log_{10} x)} \right\}}$

Solution

$$f(x) = \sqrt{\log_{10} \left\{ \frac{\log_{10} x}{2(3 - \log_{10} x)} \right\}}$$

Clearly, $f(x)$ is defined if

$$\log_{10} \left\{ \frac{\log_{10} x}{2(3 - \log_{10} x)} \right\} \geq 0, \frac{\log_{10} x}{2(3 - \log_{10} x)} > 0, \text{ and } x > 0$$

or $\frac{\log_{10} x}{2(3 - \log_{10} x)} \geq 1, \frac{\log_{10} x}{\log_{10} x - 3} < 0, \text{ and } x > 0$

or $\frac{3(\log_{10} x - 2)}{2(\log_{10} x - 3)} \leq 0, \frac{\log_{10} x}{\log_{10} x - 3} < 0, \text{ and } x > 0$

or $2 \leq \log_{10} x < 3, 0 < \log_{10} x < 3, \text{ and } x > 0$

or $10^2 \leq x < 10^3, 10^0 < x < 10^3, \text{ and } x > 0$

or $x \in [10^2, 10^3)$

Illustration 11 Find domain of $f(x) = \sin^{-1}(\log_2 x)$

Solution $f(x) = \sin^{-1}(\log_2 x)$
 Since the domain of $\sin^{-1} x$ is $[-1, 1]$, $f(x) = \sin^{-1}(\log_2 x)$ is defined if

$$-1 \leq \log_2 x \leq 1$$

or $2^{-1} \leq x \leq 2^1$

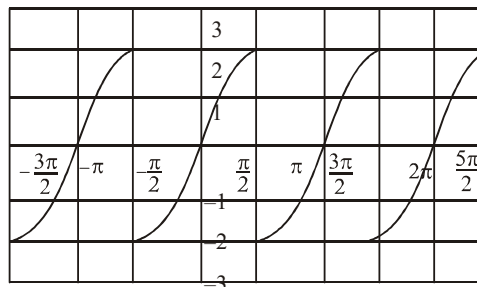
or $\frac{1}{2} \leq x \leq 2$

or domain = $\left[\frac{1}{2}, 2\right]$

Illustration 12 Find domain and range

$$f(x) = \log_{(x-4)}(x^2 - 11x + 24)$$

Solution $f(x) = \log_{(x-4)}(x^2 - 11x + 24)$
 $x - 4 > 0$ and $\neq 1$ and $x^2 - 11x + 24 > 0$
 or $x > 4$ and $\neq 5$ and $(x - 3)(x - 8) > 0$
 i.e., $x > 4$ and $\neq 5$ and $x < 3$ or $x > 8$
 or domain $(y) = (8, \infty)$



From the graph, range of the function is $(-2, 2)$.

Illustration 13 Find domain of $f(x) = \frac{3}{4 - x^2} + \log_{10}(x^3 - x)$

Solution $f(x) = \frac{3}{4 - x^2} + \log_{10}(x^3 - x)$

f is defined when
 $x \neq \pm 2$ and $x^3 - x > 0$
 or $x \neq \pm 2$ and $x(x^2 - 1) > 0$

or $x \neq \pm 2, x \in (-1, 0) \cup (1, \infty)$

or $x \in (1-, 0) \cup (1, 2) \cup (2, \infty)$

Illustration 14 Find domain of $f(x) = \frac{1}{\sqrt{\log_{1/2}(x^2 - 7x + 13)}}$

Solution $f(x) = \frac{1}{\sqrt{\log_{1/2}(x^2 - 7x + 13)}}$ exists if

$\log_{1/2}(x^2 - 7x + 13) > 0$

or $x^2 - 7x + 13 < 1$ (i)

and $x^2 - 7x + 12 < 0$ (ii)

or $x^2 - 7x + 12 < 0$ and $(x - 4)(x - 3) < 0$

or $3 < x < 4$ and $x \in \mathbb{R}$

or $3 < x < 4$

Illustration 15 Find the domain of

(a) $f(x) = \frac{1}{\sqrt{x - |x|}}$ (b) $f(x) = \frac{1}{\sqrt{x + |x|}}$

Solution

(a) $f(x) = \frac{1}{\sqrt{x - |x|}}$

$x - |x| = \begin{cases} x - x = 0, & \text{if } x \geq 0 \\ x + x = 2x, & \text{if } x < 0 \end{cases}$

or $x - |x| \leq 0$ for all x

i.e., $\frac{1}{\sqrt{x - |x|}}$ does not take real values for any $x \in \mathbb{R}$.

i.e., $f(x)$ is not defined for any $x \in \mathbb{R}$.

Hence, the domain (f) is ϕ .

(b) $f(x) = \frac{1}{\sqrt{x + |x|}}$

$x + |x| = \begin{cases} x + x, & \text{if } x \geq 0 \\ x - x, & \text{if } x < 0 \end{cases}$

or $x + |x| = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$ (i)

Now, $f(x) = \frac{1}{\sqrt{x + |x|}}$ assumes real values of $x + |x| > 0$

or $x > 0$ [using (i)]

or $x \in (0, \infty)$

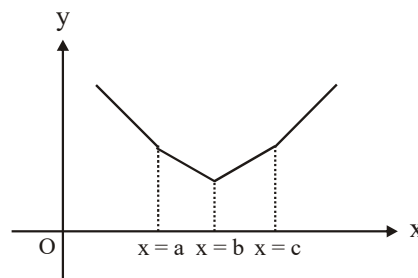
Hence, domain (f) = $(0, \infty)$

Illustration 16 If $a < b < c$, then find the range of

$f(x) = |x - a| + |x - b| + |x - c|$.

Solution $f(x)$ can be rewritten as

$$f(x) = \begin{cases} a + b + c - 3x, & x < a \\ b + c - a - x, & a \leq x < b \\ c - a - b + x, & b \leq x < c \\ 3x - a - b - c, & x \geq c \end{cases}$$



Graph of $f(x)$ is shown in the figure

Clearly, the minimum value of $f(x)$ will occur at $x = b$ which is $c - a$.

Illustration 17 Find the range of $f(x) = \sqrt{1 - \sqrt{x^2 - 6x + 9}}$

Solution

$f(x) = \sqrt{1 - \sqrt{x^2 - 6x + 9}} = \sqrt{1 - \sqrt{(x - 3)^2}} = \sqrt{1 - |x - 3|}$

Therefore, range of $f(x)$ is $[0, 1]$.

Illustration 18 Solve the following :

(a) $1 \leq |x - 2| \leq 3$ (b) $0 < |x - 3| \leq 5$

(c) $\left| \frac{x - 3}{x + 1} \right| \leq 1$

Solution

(a) $1 \leq |x - 2| \leq 3$

We know that $a \leq |x| \leq b \Leftrightarrow x \in [-b, -a] \cup [a, b]$

Given that $1 \leq |x - 2| \leq 3$

or $(x - 2) \in [-3, -1] \cup [1, 3]$

or $x \in [-1, 1] \cup [3, 5]$

(b) $0 < |x - 3| \leq 5$

or $x - 3 \neq 0$ and $|x - 3| \leq 5$

or $x \neq 3$ and $-5 \leq x - 3 \leq 5$

or $x \neq 3$ and $-2 \leq x \leq 8$

or $x \in [-2, 3) \cup (3, 8]$

(c) $\left| \frac{x - 3}{x + 1} \right| \leq 1$

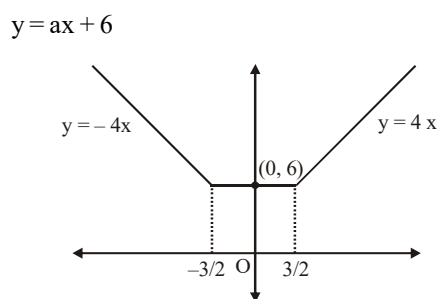
or $-1 \leq \frac{x - 3}{x + 1} \leq 1$

or $\frac{x - 3}{x + 1} - 1 \leq 0$ and $0 \leq \frac{x - 3}{x + 1} + 1$

or $\frac{-4}{x+1} \leq 0$ and $0 \leq \frac{2x-2}{x+1}$
 or $x > -1$ and $\{x < -1 \text{ or } x \geq 1\}$
 or $x \geq 1$

Illustration 19 Find the set of real value(s) of a for which the equation $|2x + 3| + |2x - 3| = ax + 6$ has more than two solutions.

Solution Given $|2x + 3| + |2x - 3| = \begin{cases} 4x, & \text{if } x \geq \frac{3}{2} \\ 6, & \text{if } -\frac{3}{2} < x < \frac{3}{2} \\ -4x, & \text{if } x \leq -\frac{3}{2} \end{cases}$ and



From the graph, it is obvious that

- if $a = 0$, we have infinite solutions in the range $[-\frac{3}{2}, \frac{3}{2}]$,
- if $0 < a < 4$ or $-4 < a < 0$, we have two solutions.
- if $a = 4$ or -4 , $x = 0$ is only solution.

Illustration 20 Find the domain of

(a) $f(x) = \frac{1}{\sqrt{x - [x]}}$ (c) $f(x) = \log\{x\}$

Solution

(a) We have $f(x) = \frac{1}{\sqrt{x - [x]}}$

We know that $0 \leq x - [x] < 1$ for all $x \in \mathbb{R}$

Also, $x - [x] = 0$ for $x \in \mathbb{Z}$

Now, $f(x) = \frac{1}{\sqrt{x - [x]}}$ is defined if

$x - [x] > 0$

or $x \in \mathbb{R} - \mathbb{Z}$

Hence, domain = $\mathbb{R} - \mathbb{Z}$

(c) $f(x) = \log\{x\}$ is defined if $\{x\} > 0$ which is true for all real numbers except integers.

Hence, the domain is $\mathbb{R} - \mathbb{Z}$.

Illustration 21 Find the range of $f(x) = \cos(\log_e\{x\})$.

Solution $f(x) = \cos(\log_e\{x\})$.

For the given function to define,

$0 < \{x\} < 1$

or $-\infty < \log_e\{x\} < 0$

For these values of $\theta = (\log_e\{x\})$, $\cos \theta$ takes all its possible values.

Hence, the range is $[-1, 1]$.

Illustration 22 Solve $(x - 2)[x] = \{x\} - 1$, (where $[x]$ and $\{x\}$

denote the greatest integer function less than equal to x and the fractional part function, respectively.

Solution for $x \geq 2$, LHS is always non-negative and RHS is always negative.

Hence, for $x \geq 2$, there is no solution.

If $1 \leq x < 2$, then $(x - 2) = (x - 1) - 1 = x - 2$, which is an identity.

For $0 \leq x < 1$, LHS is 0 and RHS is (-)ve.

So, there is no solution.

For $x < 0$, LHS is (+)ve, RHS is (-)ve.

So, there is no solution.

Hence, $x \in [1, 2)$

Illustration 23 Solve $[x]^2 - 5[x] + 6 = 0$.

Solution $[x]^2 - 5[x] + 6 = 0$

or $[x] = 2, 3$

or $x \in [2, 4)$

Illustration 24 Find the domain of $f(x) = \frac{\sqrt{1 - \sin x}}{\log_5(1 - 4x^2)}$

Solution

a. $1 - \sin x \geq 0$ or $\sin x \leq 1$ or $x \in \mathbb{R}$

b. $1 - 4x^2 > 0$ or $x \in (-1/2, 1/2)$

c. $\log_5(1 - 4x^2) \neq 0$ or $1 - 4x^2 \neq 1$ or $x \neq 0$

d. $-1 \leq 1 - \{x\} \leq 1$ or $0 \leq \{x\} \leq 2$ or $x \in \mathbb{R}$

Hence, domain is common values of a, b, c and d, i.e.,

$x \in \left(-\frac{1}{2}, \frac{1}{2}\right) - \{0\}$

Illustration 25 If $f: \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = \frac{x^2 - 4}{x^2 + 1}$, identity

the type of function.

Solution $f(x) = f(-x)$. So, f is many-one.

Also, $f(x) = 1 - \frac{5}{x^2 + 1} > 1 - 5 = -4$. So, f is into.

Illustration 26 Find the set of values of a for which the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = x^3 + (a+2)x^2 + 3ax + 5 \text{ is one-one.}$$

Solution Since $f: \mathbb{R} \rightarrow \mathbb{R}$ is one-one. Therefore, $f(x)$ is either strictly increasing or strictly decreasing.

$$\Rightarrow f'(x) > 0 \text{ or } f'(x) < 0 \text{ for all } x.$$

$$\Rightarrow 3x^2 + 2x(a+2) + 3a > 0 \text{ for all } x \in \mathbb{R}$$

$$\text{or } 3x^2 + 2x(a+2) + 3a < 0 \text{ for all } x \in \mathbb{R}$$

But, $3x^2 + 2x(a+2) + 3a$ cannot be less than zero for all $x \in \mathbb{R}$, because the curve

$y = 3x^2 + 2x(a+2) + 3a$ represents a parabola of the form $x^2 = 4ay$ for which $y > 0$ for some x .

So, $3x^2 + 2x(a+2) + 3a > 0$ for all x .

$$\Rightarrow 4(a+2)^2 - 36a < 0$$

$$[\because ax^2 + bx + c > 0 \text{ for all } x \Rightarrow D < 0]$$

$$\Rightarrow 4(a^2 + 4a + 4 - 9a) < 0$$

$$\Rightarrow (a^2 - 5a + 4) < 0$$

$$\Rightarrow (a-1)(a-4) < 0$$

$$\Rightarrow 1 < a < 4.$$

Hence, $f(x)$ is one-one if $a \in (1, 4)$.

Illustration 27 If $f: \mathbb{N} \rightarrow \mathbb{Z}$ $f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$

identify the function

Solution When n is even, let

$$f(2m_1) = f(2m_2)$$

$$\text{or } -\frac{2m_1}{2} = -\frac{2m_2}{2}$$

$$\text{or } m_1 = m_2$$

When n is odd, let

$$f(2m_1 + 1) = f(2m_2 + 1)$$

$$\text{or } \frac{2m_1 + 1 - 1}{2} = \frac{2m_2 + 1 - 1}{2} \text{ or } m_1 = m_2$$

Therefore, $f(x)$ is one-one.

$$\text{Also, when } n \text{ is even, } -\frac{n}{2} = -\frac{2m}{2} = -m$$

$$\text{When } n \text{ is odd, } \frac{n-1}{2} = \frac{2m+1-1}{2} = m$$

Hence, the range of the function is \mathbb{Z} .

Therefore, function is onto.

Illustration 28 Let C and \mathbb{R} denote the sets of all complex numbers and all real numbers respectively. Then show that $f: C \rightarrow \mathbb{R}$ given by $f(z) = |z|$ for all $z \in C$ is neither one-one nor onto.

Solution We find that $z_1 = 1 - i$ and $z_2 = 1 + i$ are two distinct complex numbers such that $|z_1| = |z_2|$

$$\text{i.e., } z_1 \neq z_2 \text{ but, } f(z_1) = f(z_2)$$

This shows that different elements may have the same image.

So, f is not an injection.

Surjectivity : f is not a surjection, because negative real numbers in \mathbb{R} do not have their pre-images in C . In other words, for every negative real number a there is no complex number $z \in C$ such that $f(z) = |z| = a$. So, f is not a surjection.

Illustration 29 If $f(x + f(y)) = f(x) + y \forall x, y \in \mathbb{R}$ and $f(0) = 1$, then find the value of $f(7)$.

Solution $f(x + f(y)) = f(x) + y, f(0) = 1$

Putting $y = 0$, we get

$$f(x + f(0)) = f(x) + 0$$

$$\text{or } f(x + 1) = f(x) \forall x \in \mathbb{R}$$

Thus, $f(x)$ is periodic with 1 as one of its period. Hence,

$$f(7) = f(6) = f(5) = \dots = f(1) = f(0) = 1$$

Illustration 30 If $f: \mathbb{R}^+ \rightarrow \mathbb{R}$, $f(x) + 3xf\left(\frac{1}{x}\right) = 2(x+1)$, then find $f(x)$.

$$\text{Solution } f(x) + 3x f\left(\frac{1}{x}\right) = 2(x+1) \quad \dots(i)$$

Replacing x by $\frac{1}{x}$, we get

$$f\left(\frac{1}{x}\right) + 3\frac{1}{x}f(x) = 2\left(\frac{1}{x} + 1\right)$$

$$\text{or } xf\left(\frac{1}{x}\right) + 3f(x) = 2(x+1) \quad \dots(ii)$$

From (i) and (ii), we have $f(x) = \frac{x+1}{2}$.

Illustration 31 Consider $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $f(3) = 1$ for

$$a \in \mathbb{R}^+ \text{ and } f(x) \cdot f(y) + f\left(\frac{3}{x}\right)f\left(\frac{3}{y}\right) = 2f(xy) \forall x, y \in \mathbb{R}^+.$$

Then find $f(x)$.

Solution $f(x) \cdot f(y) + f\left(\frac{3}{x}\right)f\left(\frac{3}{y}\right) = 2f(xy)$

Put $x = y = 1$.

Then, $f^2(1) + f^2(3) = 2f(1)$

or $(f(1) - 1)^2 = 0$ or $f(1) = 1$

Now, put $y = 1$. Then

$$f(x)f(1) + f\left(\frac{3}{x}\right)f(3) = 2f(x)$$

or $f(x) = f\left(\frac{3}{x}\right) \forall x > 0$ (i)

or $f(x)f\left(\frac{3}{x}\right) + f\left(\frac{3}{x}\right)f(x) = 2f(3)$

or $f(x)f\left(\frac{3}{x}\right) = 1$ (ii)

Therefore, from (i) and (ii),

$$f^2 = 1 \forall x > 0$$

Put $x = y = \sqrt{t}$. Then,

$$f^2(\sqrt{t}) + f^2\left(\frac{3}{\sqrt{t}}\right) = 2f(t) \text{ or } f(t) > 0$$

$\therefore f(x) = 1 \forall x > 0$

Illustration 32 Determine the function satisfying

$$f^2(x + y) = f^2(x) + f^2(y) \forall x, y \in \mathbb{R}.$$

Solution Given $f^2(x + y) = f^2(x) + f^2(y)$

Put $x = y = 0$.

Therefore, $f^2(0) = f^2(0) + f^2(0)$

$\therefore f(0) = 0$

Now, replace y by $-x$.

Therefore, $f^2(0) = f^2(x) + f^2(-x)$

or $f^2(x) + f^2(-x) = 0$

$\therefore f(x) = 0$

Illustration 33 If $f(x + 2a) = f(x - 2a)$, then prove that $f(x)$ is periodic.

Solution $f(x + 2a) = f(x - 2a)$

Replacing x by $x + 2a$, we get

$$f(x) = f(x + 4a)$$

Therefore, $f(x)$ is periodic with period $4a$.

Illustration 34 $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x^2 + x + 3) + 2f(x^2 - 3x + 5) = 6x^2 - 10x + 17 \forall x \in \mathbb{R}$, then find the function $f(x)$.

Solution Obviously, f is linear polynomial.

Let $f(x) = ax + b$.

Hence, $f(x^2 + x + 3) + 2f(x^2 - 3x + 5) \equiv 6x^2 - 10x + 17$

or $[a(x^2 + x + 3) + b] + 2[a(x^2 - 3x + 5) + b] \equiv 6x^2 - 10x + 17$

or $a + 2a = 6$ (i)

and $a - 6a = -10$ (ii)

or $a = 2$

(comparing coeff. of x^2 and coeff. of x both sides)

Again, $3a + b + 10a + 2b = 17$

or $6 + b + 20 + 2b = 17$

$\therefore b = -3$

or $f(x) = 2x - 3$

or $f(5) = 7$

Illustration 35 If $f(a - x) = f(a + x)$ and $f(b - x) = f(b + x)$ for all real x , where $a, b (a > b)$ are constants, then prove that $f(x)$ is a periodic function.

Solution $f(x) = f(b + (x - b))$

$$= f(b - (x - b))$$

$$= f(2b - x)$$

$$= f(a + (2b - x - a))$$

$$= f(a - (2b - x - a))$$

$$= f(2a - 2b + x)$$

Hence, $f(x)$ is periodic with period $2a - 2b$.

Illustration 36 If f is the greatest integer function and g is the modulus function, then find the value of

$$(g \circ f)\left(-\frac{5}{3}\right) - (f \circ g)\left(-\frac{5}{3}\right).$$

Solution Given $(g \circ f)\left(-\frac{5}{3}\right) - (f \circ g)\left(-\frac{5}{3}\right)$.

$$= g\left\{f\left(-\frac{5}{3}\right)\right\} - f\left\{g\left(-\frac{5}{3}\right)\right\} = g(-2) - f\left(\frac{5}{3}\right) = 2 - 1 = 1$$

Illustration 37 If the domain of $y = f(x)$ is $[-3, 2]$, then find the domain of $g(x) = f(|[x]|)$, where $[]$ denotes the greatest integer function.

Solution Here, $f(x)$ is defined by $[-3, 2]$.

So, $x \in [-3, 2]$

For $g(x) = f(|[x]|)$ to be defined, we must have

$$-3 \leq |[x]| \leq 2$$

or $0 \leq [x] \leq 2$ [As $|x| \geq 0$ for all x]

or $-2 \leq [x] \leq 2$ [As $|x| \leq a \Rightarrow -a \leq x \leq a$]

or $-2 \leq x < 3$ [By the definition of greatest integral function]

Hence, domain of $g(x)$ is $[-2, 3)$.

Illustration 38 If the functions f and g are given by

$$f = \{(1, 2), (3, 5), (4, 1)\}$$

$$\text{and } g = \{(2, 3), (5, 1), (1, 3)\},$$

find the range of f and g . Also, write down $f \circ g$ and $g \circ f$ as sets of ordered pairs.

Solution We have

$$\begin{aligned} \text{Range of } f &= \text{Set of second components of ordered pairs in} \\ &= \{2, 5, 1\} \end{aligned}$$

Similarly,

$$\text{Range of } g = \{3, 1\}$$

$$\text{We have, domain } f = \{1, 3, 4\}, \text{ domain } g = \{2, 5, 1\}$$

Clearly, $\text{range } f \subset \text{domain } g$ and $\text{range } g \subset \text{domain } f$.

So, $f \circ g$ and $g \circ f$ both exist.

$$\text{Now, } f \circ g(2) = f(g(2)) = f(3) = 5;$$

$$f \circ g(5) = f(g(5)) = f(1) = 2;$$

$$\text{and } f \circ g(1) = f(g(1)) = g(3) = 5$$

$$\therefore f \circ g = \{(2, 5), (5, 2), (1, 5)\}$$

$$\text{Also, } g \circ f(1) = g(f(1)) = g(2) = 3;$$

$$g \circ f(3) = g(f(3)) = g(5) = 1$$

$$\text{and, } g \circ f(4) = g(f(4)) = g(1) = 3$$

$$\therefore g \circ f = \{(1, 3), (3, 1), (4, 3)\}$$

Illustration 39 If the function $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2 + 2$

$$\text{and } g: \mathbb{R} \rightarrow \mathbb{R} \text{ be given by } g(x) = \frac{x}{x-1}$$

Find $f \circ g$ and $g \circ f$.

Solution Clearly, $\text{range } f = \text{domain } g$

and, $\text{range } g = \text{domain } f$.

So $f \circ g$ and $g \circ f$ both exist.

Now,

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^2 + 2 = \frac{x^2}{(x-1)^2} + 2$$

$$\text{and } (g \circ f)(x) = g(f(x)) = g(x^2 + 2) = \frac{x^2 + 2}{(x^2 + 2) - 1} = \frac{x^2 + 2}{x^2 + 1}$$

Hence, $f \circ g: \mathbb{R} \rightarrow \mathbb{R}$ and $g \circ f: \mathbb{R} \rightarrow \mathbb{R}$ are given by

$$(f \circ g)(x) = \frac{x^2 + 2}{x^2 + 1} \text{ and, } (g \circ f)(x) = \frac{x^2}{(x-1)^2} + 2$$

respectively

Illustration 40 If $f(x) = \begin{cases} x^2, & \text{for } x \geq 0 \\ x, & \text{for } x < 0 \end{cases}$, then $f \circ f(x)$ is given

by

(A) x^2 for $x \geq 0$, x for $x < 0$ (B) x^4 for $x \geq 0$, x^2 for $x < 0$

(C) x^4 for $x \geq 0$, $-x^2$ for $x < 0$ (D) x^4 for $x \geq 0$, x for $x < 0$

Solution (D) $f(f(x)) = \begin{cases} (f(x))^2, & \text{for } f(x) \geq 0 \\ f(x), & \text{for } f(x) < 0 \end{cases}$

$$= \begin{cases} (x^2)^2, & x^2 \geq 0, & x \geq 0 \\ x^2, & x \geq 0, & x < 0 \\ x^2, & x^2 < 0, & x \geq 0 \\ x, & x < 0, & x < 0 \end{cases} = \begin{cases} x^4, & x \geq 0 \\ x, & x < 0 \end{cases}$$

Illustration 41 If $f(x) = \begin{cases} \log_e x, & 0 < x < 1 \\ x^2 - 1, & x \geq 1 \end{cases}$

$$\text{and } g(x) = \begin{cases} x+1, & x < 2 \\ x^2 - 1, & x \geq 2 \end{cases}, \text{ then find } g(f(x)).$$

Solution $f(x) = \begin{cases} \log_e x, & 0 < x < 1 \\ x^2 - 1, & x \geq 1 \end{cases}$ and $g(x) = \begin{cases} x+1, & x < 2 \\ x^2 - 1, & x \geq 2 \end{cases}$.

$$g(f(x)) = \begin{cases} f(x)+1, & f(x) < 2 \\ (f(x))^2 - 1, & f(x) \geq 2 \end{cases}$$

$$= \begin{cases} \log_e x + 1, & \log_e x < 2, & 0 < x < 1 \\ x^2 - 1 + 1, & x^2 - 1 < 2, & x \geq 1 \\ (\log_e x)^2 - 1, & \log_e x \geq 2, & 0 < x < 1 \\ (x^2 - 1)^2 - 1, & x^2 - 1 \geq 2, & x \geq 1 \end{cases}$$

$$= \begin{cases} \log_e x + 1, & x < e^2, & 0 < x < 1 \\ x^2 - 1 + 1, & -\sqrt{3} < x < \sqrt{3}, & x \geq 1 \\ (\log_e x)^2 - 1, & x \geq e^2, & 0 < x < 1 \\ (x^2 - 1)^2 - 1 & x \leq -\sqrt{3} \text{ or } x \geq \sqrt{3}, & x \geq 1 \end{cases}$$

$$= \begin{cases} \log_e x + 1, & 0 < x < 1 \\ x^2, & 1 \leq x < \sqrt{3} \\ (x^2 - 1)^2 - 1, & x \geq \sqrt{3} \end{cases}$$

Illustration 42 If $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(x) = x^2 - 3x + 2, \text{ find } f(f(x)).$$

Solution We have,

$$\begin{aligned} f(f(x)) &= f(x^2 - 3x + 2) \\ &= f(y), \text{ where } y = x^2 - 3x + 2. \\ &= y^2 - 3y + 2 \\ &= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2 \\ &= x^4 - 6x^3 + 10x^2 - 3x \end{aligned}$$

Illustration 43 If $f(x) =$

$$\begin{cases} x^3 + 1, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}, g(x) = \begin{cases} (x-1)^{1/3}, & x < 1 \\ (x-1)^{1/2}, & x \geq 1 \end{cases} \text{ compute } g \circ f(x).$$

Solution We have, $g \circ f(x) = g(f(x)) = \begin{cases} g(x^3 + 1), & x < 0 \\ g(x^3 + 1)^{1/3}, & x \geq 0 \end{cases}$

$$= \begin{cases} (x^3 + 1 - 1)^{1/3} & x < 0 \\ (x^2 + 1 - 1)^{1/2} & x \geq 0 \end{cases}$$

$$= \begin{cases} x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

= x for all x.

Hence, $\text{gof}(x) = x$ for all x.

Illustration 44 Identify the following functions whether odd or even or neither :

(i) $f(x) = \log\left(\frac{x^4 + x^2 + 1}{x^2 + x + 1}\right)$

Solution $\log\left(\frac{x^4 + x^2 + 1}{x^2 + x + 1}\right) = \log(x^2 - x + 1)$,

which is neither odd nor even.

(ii) $f(x) = \log\left(x + \sqrt{x^2 + 1}\right)$

Solution $f(x) = \log\left(x + \sqrt{x^2 + 1}\right)$

$$f(-x) = \log\left(-x + \sqrt{x^2 + 1}\right)$$

or $f(x) + f(-x) = \log\left(x + \sqrt{x^2 + 1}\right) + \log\left(-x + \sqrt{x^2 + 1}\right)$

$$= \log\left(\sqrt{x^2 + 1} + x\right) + \log\left(\sqrt{x^2 + 1} - x\right)$$

$$= \log(x^2 + 1 - x^2) = \log 1 = 0$$

or $f(-x) = -f(x)$

Hence, $f(x)$ is an odd function.

(iii) $f(x) = \begin{cases} x|x| & , \quad x \leq -1 \\ [x+1] + [1-x] & , \quad -1 < x < 1 \\ -x|x| & , \quad x \geq 1 \end{cases}$

where [] represents the greatest integer function.

Solution $f(-x) = \begin{cases} -x|-x| & , \quad -x \leq -1 \\ [-x+1] + [1+x] & , \quad -1 < -x < 1 \\ -(-x)|-x| & , \quad -x \geq 1 \end{cases}$

$$= \begin{cases} -x|x| & , \quad x \geq 1 \\ [1-x] + [1+x] & , \quad -1 < x < 1 \\ x|x| & , \quad x \leq 1 \end{cases}$$

$$= f(x)$$

Hence, the function is even.

Illustration 45 If $g : [-2, 2] \rightarrow \mathbb{R}$, where $g(x) = x^3 + \tan x +$

$\left[\frac{x^2 + 1}{P}\right]$ is an odd function, then the value of parametric

P, where [.] denotes the greatest integer function, is

(A) $-5 < P < 5$

(B) $P < 5$

(C) $P > 5$

(D) none of these

Solution (C) $g(x) = x^3 + \tan x + \left[\frac{x^2 + 1}{P}\right]$

or $g(-x) = (-x)^3 + \tan(-x) + \left[\frac{(-x)^2 + 1}{P}\right]$

$$= -x^3 - \tan x + \left[\frac{x^2 + 1}{P}\right]$$

or $g(x) = g(-x) = 0$

Because $g(x)$ is a odd function,

$$\left(-x^3 - \tan x + \left[\frac{x^2 + 1}{P}\right]\right) + \left(\frac{-x^3 - \tan x}{+\left[\frac{x^2 + 1}{P}\right]}\right) = 0$$

or $2\left[\frac{x^2 + 1}{P}\right] = 0$ or $0 \leq \frac{x^2 + 1}{P} < 1$

Now, $x \in [-2, 2]$

$\therefore 0 \leq \frac{5}{P} < 1$ or $P > 5$

Illustration 46 $f(x) = \frac{\cos x}{\left[\frac{2x}{\pi}\right] + \frac{1}{2}}$, where x is not an integral

multiple of π and [.] denotes the greatest integer function, is

(A) an odd function

(B) an even function

(C) neither odd nor even

(D) none of these

Solution (A) $f(-x) = \frac{\cos(-x)}{\left[\frac{-2x}{\pi}\right] + \frac{1}{2}} = \frac{\cos x}{-1 - \left[\frac{2x}{\pi}\right] + \frac{1}{2}}$

(As x is not an integral multiple of π)

$$= -\frac{\cos(-x)}{\left[\frac{-2x}{\pi}\right] + \frac{1}{2}} = -f(x)$$

Illustration 47 Find the inverse of the following functions :

(A) $f : \mathbb{R} \rightarrow (-\infty, 1)$ given by $f(x) = 1 - 2^{-x}$

Solution Let $y = 1 - 2^{-x}$

or $2^{-x} = 1 - y$

or $-x = \log_2(1 - y)$

or $f^{-1}(x) = g(x) = -\log_2(1-x)$

(B) $f: Z \rightarrow Z$ defined by $f(x) = [x + 1]$, where $[.]$ denotes the greatest integer function.

Solution Since the domain of the function is I , we have

$f(x) = x + 1$

or $f^{-1}(x) = x - 1$.

(C) $f(x) = \begin{cases} x^3 - 1, & x < 2 \\ x^2 + 3, & x \geq 2 \end{cases}$

Solution $f(x) = \begin{cases} x^3 - 1, & x < 2 \\ x^2 + 3, & x \geq 2 \end{cases}$

For $f(x) = x^3 - 1, x < 2, f^{-1}(x) = (x + 1)^{1/3}, x < 7$
 (As $x < 2 \Rightarrow x^3 < 8 \Rightarrow x^3 - 1 < 7$)

For $f(x) = x^2 + 3, x \geq 2, f^{-1}(x) = (x - 3)^{1/2}, x \geq 7$
 (As $x \geq 2 \Rightarrow x^2 \geq 4 \Rightarrow x^2 + 3 \geq 7$)

Hence, $f^{-1}(x) = \begin{cases} (x + 1)^{1/3}, & x < 7 \\ (x - 3)^{1/2}, & x \geq 7 \end{cases}$

(D) $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 2$

Solution $y = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 2$
 $= \frac{e^{2x} - 1}{e^{2x} + 1} + 2$

or $e^{2x} = \frac{1-y}{y-3} = \frac{y-1}{3-y}$

or $x = \frac{1}{2} \log_e \left(\frac{y-1}{3-y} \right)$

or $f^{-1}(y) = \log_e \left(\frac{y-1}{3-y} \right)^{1/2}$

or $f^{-1}(x) = \log_e \left(\frac{x-1}{3-x} \right)^{1/2}$

Illustration 48 Let $f: \left[-\frac{\pi}{3}, \frac{2\pi}{3}\right] \rightarrow [0, 4]$ be a function

defined as $f(x) = \sqrt{3} \sin x - \cos x + 2$. Then $f^{-1}(x)$ is given by

(A) $\sin^{-1} \left(\frac{x-2}{2} \right) - \frac{\pi}{6}$ (B) $\sin^{-1} \left(\frac{x-2}{2} \right) + \frac{\pi}{6}$

(C) $\frac{2\pi}{3} + \cos^{-1} \left(\frac{x-2}{2} \right)$ (D) None of these

Solution (B) $y = f(x) = \sqrt{3} \sin x - \cos x + 2 = 2 \sin \left(x - \frac{\pi}{6} \right) + 2$
(i)

Since $f(x)$ is one-one and onto, f is invertible.

From (i), $\sin \left(x - \frac{\pi}{6} \right) = \frac{y-2}{2}$

or $x = \sin^{-1} \frac{y-2}{2} + \frac{\pi}{6}$

or $f^{-1}(x) = \sin^{-1} \left(\frac{x-2}{2} \right) + \frac{\pi}{6}$

Illustration 49 Which of the following functions is the inverse of itself?

- (A) $f(x) = \frac{1-x}{1+x}$ (B) $f(x) = 5^{\log x}$
 (C) $f(x) = 2^{x(x-1)}$ (D) None of these

Solution (A) By checking for different functions, we find that for

$f(x) = \frac{1-x}{1+x}, f^{-1}(x) = f(x)$.

Illustration 50 If the function $f: [1, \infty) \rightarrow [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is

- (A) $\left(\frac{1}{2}\right)^{x(x-1)}$ (B) $\frac{1}{2}(1 + \sqrt{1 + 4 \log_2 x})$
 (C) $\frac{1}{2}(1 - \sqrt{1 + 4 \log_2 x})$ (D) Not defined

Solution (B) Given $y = 2^{x(x-1)}$

or $x(x-1) = \log_2 y$

or $x^2 - x - \log_2 y = 0$

or $x = \frac{1 \pm \sqrt{1 + 4 \log_2 y}}{2}$

Only $x = \frac{1 + \sqrt{1 + 4 \log_2 y}}{2}$ lies in the domain. So,

$f^{-1}(x) = \frac{1}{2} [1 + \sqrt{1 + 4 \log_2 x}]$

Illustration 51 Discuss whether the function $f(x) = \sin(\cos x + x)$ is periodic or not. If yes, then what is its period?

Solution Clearly,

$f(x + 2\pi) = \sin \{ \cos(2\pi + x) + 2\pi + x \}$
 $= \sin \{ 2\pi + (x + \cos x) \}$
 $= \sin(x + \cos x)$

Hence, period is 2π .

Illustration 52 Let $f(x, y)$ be a periodic function satisfying

$$f(x, y) = f(2x + 2y, 2y - 2x) \text{ for all } x, y \in \mathbb{R}$$

Define $g(x) = f(2^x, 0)$. Show that $g(x)$ is a periodic function with period 12.

Solution We have,

$$f(x, y) = f(2x + 2y, 2y - 2x) \text{ for all } x, y \in \mathbb{R} \quad \dots(i)$$

$$f(x, y) = f(2(2x + 2y) + 2(2y - 2x), 2(2y - 2x) - 2(2x + 2y))$$

$$f(x, y) = f(8y, -8x)$$

$$= f(8 \times -8x, -8 \times 8y)$$

$$f(x, y) = f(-64x, -64y)$$

$$= f(-64 \times -64x, -64 \times -64y)$$

$$= f(2^{12}x, 2^{12}y)$$

$$f(x, 0) = f(2^{12}x, 0)$$

$$f(2^y, 0) = f(2^{12} \times 2^y, 0)$$

$$f(2^y, 0) = f(2^{12+y}, 0)$$

$$f(2^x, 0) = f(2^{12+x}, 0)$$

$$g(x) = g(12 + x) \quad \text{for all } x$$

Hence, $g(x)$ is a periodic function with period 12.

Illustration 53 If $f(x) = \lambda|\sin x| + \lambda^2|\cos x| + g(\lambda)$ has period equal to $\pi/2$, then find the value of λ .

Solution Since the period of $|\sin x| + |\cos x|$ is $\pi/2$, it is possible when $\lambda = 1$.

Illustration 54 If the period of $\frac{\cos(\sin nx)}{\tan\left(\frac{x}{n}\right)}$, $n \in \mathbb{N}$ is 6π then

$n =$

(A) 3

(B) 2

(C) 6

(D) 1

Solution (C) The period of $\cos(\sin nx)$ is $\frac{\pi}{n}$ and the period of

$$\tan\left(\frac{x}{n}\right) \text{ is } \pi n.$$

$$\text{Thus, } 6\pi = \text{LCM}\left(\frac{\pi}{n}, \pi n\right)$$

$$\text{or } 6\pi = \frac{\pi}{n}\lambda_1 \text{ or } n = \frac{\lambda_1}{6}, \text{ and } 6\pi = \lambda_2 \pi n$$

$$\text{or } n = \frac{6}{\lambda_2}, \lambda_1, \lambda_2 \in \mathbb{I}^+$$

$$\text{From } n = \frac{6}{\lambda_2}, n = 6, 3, 2, 1.$$

Clearly, for $n = 6$, we get the period of $f(x)$ to be 6π .

Illustration 55 Find the period if $f(x) = \sin x + \{x\}$, where $\{x\}$ is the fractional part of x .

Solution Here, $\sin x$ is periodic with period 2π and $\{x\}$ is periodic with 1. The LCM of 2π (irrational) and 1 (rational) does not exist.

Thus, $f(x)$ is not periodic.

Illustration 56 Find the period of $f(x) = \sin \frac{2\pi x}{3} + \cos \frac{\pi x}{2}$.

Solution Since $\sin \frac{2\pi}{3}x$ and $\cos \frac{\pi}{2}x$ are periodic functions with periods $\frac{2\pi}{2\pi/3} = 3$ and $\frac{2\pi}{\pi/2} = 4$ respectively. Therefore, $f(x)$ is periodic with period equal to LCM of 3 and 4 i.e., 12.

Illustration 57 Match the column

Column I

Column II

(P) $f(x) = \sin^3 x + \cos^4 x$

a. $\pi/2$

(Q) $f(x) = \cos^4 x + \sin^4 x$

b. π

(R) $f(x) = \sin^3 x + \cos^3 x$

c. 2π

(S) $f(x) = \cos^4 x - \sin^4 x$

Solution

(P) $f(x) = \sin^3 x + \cos^4 x$,

$\sin^3 x$ has period 2π and $\cos^4 x$ has period π , and L.C.M. of π and 2π is 2π .

Hence, period is 2π .

(Q) $f(x) = \sin^4 x + \cos^4 x$

Both $\sin^4 x$ and $\cos^4 x$ have the same period π , and L.C.M of π and π is π .

But, $f(x + \pi/2) = f(x)$.

Then, period is $\pi/2$

(R) Both $\sin^3 x$ and $\cos^3 x$ has the same period 2π , and L.C.M of 2π and 2π is 2π .

Hence, period is 2π , $[f(x + \pi) \neq f(x)]$.

(S) $f(x) = \cos^4 x - \sin^4 x$

Both $\sin^4 x$ and $\cos^4 x$ have the same period π , and L.C.M of π and π is π .

Hence, period is π $[f(x + \pi/2) \neq f(x)]$.

Illustration 58 Which of the following functions is not periodic ?

(A) $|\sin 3x| + \sin^2 x$

(B) $\cos \sqrt{x} + \cos^2 x$

(C) $\cos 4x + \tan^4 x$

(D) $\cos 2x + \sin x$

Solution (B)

Since $\cos \sqrt{x}$ is not periodic, $\cos \sqrt{x} + \cos^2 x$ is not periodic although $\cos^2 x$ is periodic.

Illustration 59 Find the period of

- (a) $\frac{|\sin 4x| + |\cos 4x|}{|\sin 4x - \cos 4x| + |\sin 4x + \cos 4x|}$
- (b) $f(x) = \sin \frac{\pi x}{n!} - \cos \frac{\pi x}{(n+1)!}$
- (c) $f(x) = \sin x + \tan \frac{x}{2} + \sin \frac{x}{2^2} + \tan \frac{x}{2^3} + \dots + \sin \frac{x}{2^{n-1}} + \tan \frac{x}{2^n}$

Solution

- (a) Period of $|\sin 4x| + |\cos 4x|$ is $\frac{\pi}{8}$

Period of $|\sin 4x - \cos 4x| + |\sin 4x + \cos 4x| = \frac{\pi}{8}$

Because period of $|\sin x - \cos x| + |\sin x + \cos x| = \frac{\pi}{8}$

the period of given function is $\frac{\pi}{8}$

- (b) $f(x) = \sin \frac{\pi x}{n!} - \cos \frac{\pi x}{(n+1)!}$

Period of $\sin \frac{\pi x}{n!}$ is $\frac{2\pi}{\left(\frac{\pi}{n!}\right)} = 2n!$ and period of $\cos \frac{\pi x}{(n+1)!}$

is $\frac{2\pi}{\frac{\pi}{(n+1)!}} = 2(n+1)!$

Hence, period of $f(x) = \text{L.C.M. of } \{2n!, 2(n+1)!\} = 2(n+1)!$

- (c) $f(x) = \sin x + \tan \frac{x}{2} + \sin \frac{x}{2^2} + \tan \frac{x}{2^3} + \dots + \sin \frac{x}{2^{n-1}} + \tan \frac{x}{2^n}$

Period of $\sin x$ is 2π .

Period of $\tan \frac{x}{2}$ is 2π .

Period of $\sin \frac{x}{2^2}$ is 8π .

Period of $\tan \frac{x}{2^3}$ is 8π .

Period of $\tan \frac{x}{2^n}$ is $2^n\pi$.

Hence, period of $f(x) = \text{L.C.M. of } (2\pi, 8\pi, \dots, 2^n\pi) = 2^n\pi$

Draw the graph of the following functions (60 to 64)

Illustration 60 $f(x) = ||x-2|-3|$.

Solution

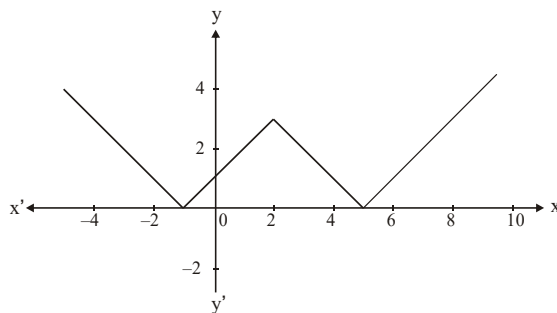


Illustration 61 $f(x) = \tan x$.

Solution

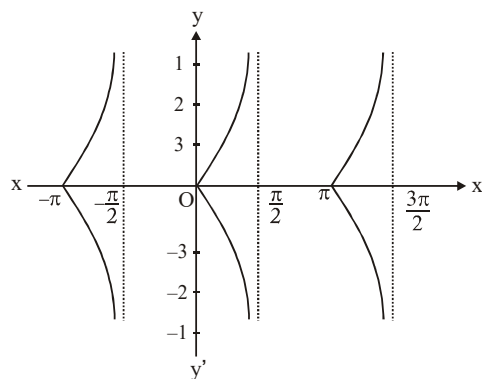


Illustration 62 $f(x) = |x^2 - 3|x| + 2|$.

Solution

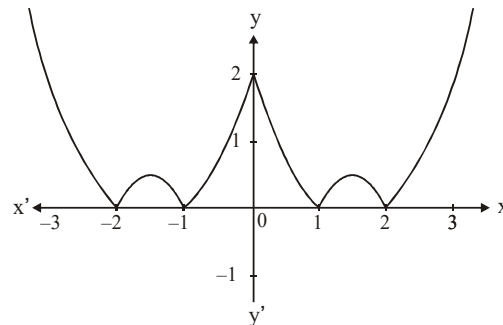


Illustration 63 $f(x) = -|x-1|^{1/2}$.

Solution

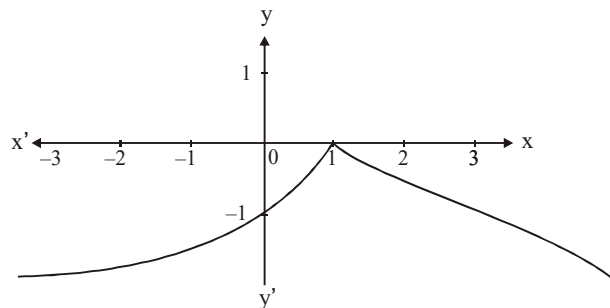
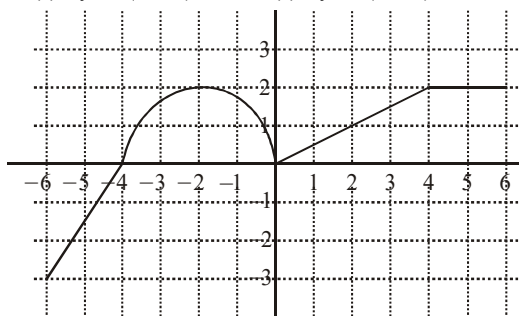


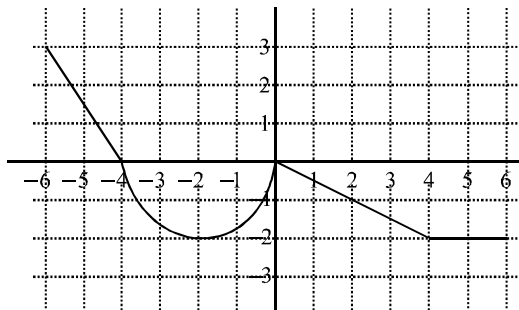
Illustration 64 Given the graph of $f(x)$, graph each one of the following functions :

- Solution (a)** $y = f(x) + 3$ **(b)** $y = -f(x-1)$

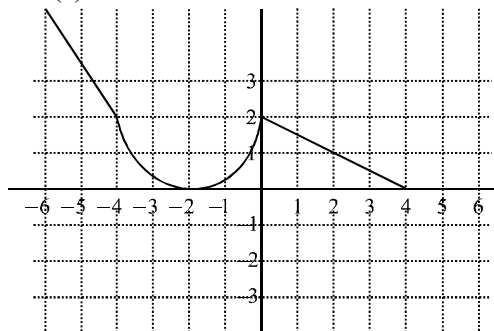
(c) $y = f(x+1) - 2$ (d) $y = f(1-x)$



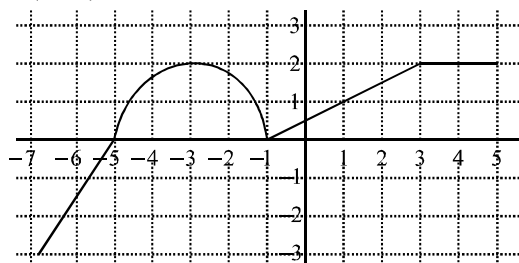
- a. $y = f(x) + 3$
- b. $y = -f(x-1)$



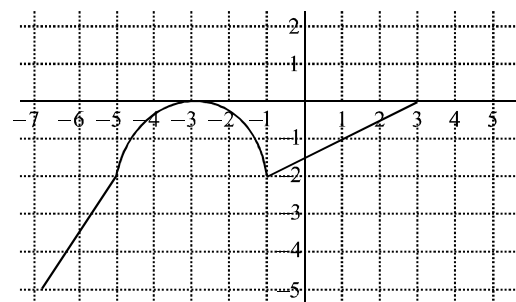
Now, shift the above graph 2 units upward to get $y = 2 - f(x)$.



c. $y = f(x+1) - 2$



Now, shift the graph of $y = f(x)$, 1 units left to get $y = f(x+1)$



d. $y = f(1-x)$

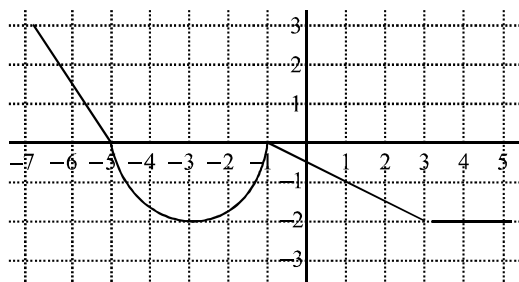


Illustration 65 Which of the following pair(s) of functions have the same graphs ?

- (a) $f(x) = \frac{\sec x}{\cos x} - \frac{\tan x}{\cot x}$, $g(x) = \frac{\cos x}{\sec x} + \frac{\sin x}{\operatorname{cosec} x}$
- (b) $f(x) = \operatorname{sgn}(x^2 - 6x + 10)$,
 $g(x) = \operatorname{sgn}\left(\cos^2 x + \sin^2\left(x + \frac{\pi}{3}\right)\right)$, where sgn denotes signum function.
- (c) $f(x) = e^{(\ln x^2 + 3x + 3)}$, $g(x) = x^2 + 3x + 3$

(d) $f(x) = \frac{\sin x}{\sec x} + \frac{\cos x}{\operatorname{cosec} x}$, $g(x) = \frac{2 \cos^2 x}{\cot x}$

Solution (a) Domain of both

$$f(x) = \frac{\sec x}{\cos x} - \frac{\tan x}{\cot x}, g(x) = \frac{\cos x}{\sec x} + \frac{\sin x}{\operatorname{cosec} x} \text{ is}$$

$$x \in \mathbb{R} - \left\{ \frac{n\pi}{2}, n \in \mathbb{Z} \right\}$$

Also, both functions simplify to 1.
Hence, both functions are identical.

(b) As $x^2 - 6x + 10 = (x-3)^2 + 1 > 0$
 $f(x) = 1 \forall x \in \mathbb{R}$

Also, $\cos^2 x + \sin^2\left(x + \frac{\pi}{6}\right) > 0$

Hence, $g(x) = 1 \forall x \in \mathbb{R}$
 Therefore, $f(x)$ and $g(x)$ are identical

(c) $f(x) = e^{\ln(x^2 + 3x + 3)}$
 As $x^2 + 3x + 3 = \left(x + \frac{3}{2}\right)^2 + \frac{3}{4} > 0 \forall x \in \mathbb{R}$

$f(x) = x^2 + 3x + 3 \forall x \in \mathbb{R}$
 Therefore, $f(x)$ is identical to $g(x)$.

(d) $f(x) = \frac{\sin x}{\sec x} + \frac{\cos x}{\operatorname{cosec} x}$
 $= 2 \sin x \cos x$
 $= \frac{2 \cos^2 x}{\cot x}$
 $= g(x)$

Also, domain of both the functions is $x \in \mathbb{R} - \left\{ \frac{n\pi}{2}, n \in \mathbb{Z} \right\}$

EXERCISE I

1. If $[.]$ denotes the greatest integer function, then the value

$$\text{of } \sum_{r=1}^{100} \left[\frac{1}{2} + \frac{r}{100} \right] \text{ is}$$

- (A) 49 (B) 50
(C) 51 (D) 52

2. The function $f(x) = \int_0^x \log_e \left(\frac{1-x}{1+x} \right) dx$ is

- (A) An even function (B) An odd function
(C) A periodic function (D) None of these

3. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = \frac{x^2 + 2x + 5}{x^2 + x + 1}$ is

- (A) One-one and into (B) One-one and onto
(C) Many one and onto (D) Many one and into

4. Let the function $f(x) = x^2 + x + \sin x - \cos x + \log(1 + |x|)$ be defined over the interval $[0, 1]$. The odd extension of $f(x)$ in the interval $[-1, 1]$ is

- (A) $x^2 + x + \sin x + \cos x - \log(1 + |x|)$
(B) $-x^2 + x + \sin x + \cos x - \log(1 + |x|)$
(C) $-x^2 + x + \sin x - \cos x + \log(1 + |x|)$
(D) None of the above

5. If $f: \mathbb{R} \rightarrow \mathbb{R}$, $g: \mathbb{R} \rightarrow \mathbb{R}$ be two given functions, the $f(x) = 2 \min(f(x) - g(x), 0)$ equals

- (A) $f(x) + g(x) - |g(x) - f(x)|$ (B) $f(x) + g(x) + |g(x) - f(x)|$
(C) $f(x) - g(x) + |g(x) - f(x)|$ (D) $f(x) - g(x) - |g(x) - f(x)|$

6. If $f: [-4, 0] \rightarrow \mathbb{R}$ is defined by $e^x + \sin x$, its even extension to $[-4, 4]$ is given by

- (A) $-e^{-|x|} - \sin |x|$ (B) $e^{-|x|} - \sin |x|$
(C) $e^{-|x|} + \sin |x|$ (D) $-e^{-|x|} + \sin |x|$

7. Let $A = \{1, 2, 3, 4, 5, 6\}$. If f be a bijective function from A to A , then the number of such functions for which $f(\lambda) \neq \lambda$, $\lambda = 1, 2, 3, 4, 5, 6$ is

- (A) 44 (B) 265
(C) 325 (D) 4585

8. The function $f(x) = \sin\left(\frac{\pi x}{n!}\right) - \cos\left(\frac{\pi x}{(n+1)!}\right)$ is

- (A) Non-periodic
(B) Periodic, with period $2(n!)$
(C) Periodic, with period $2(n+1)!$
(D) None of the above

9. If $f(x)$ is a polynomial function of the second degree such that $f(-3) = 6$, $f(0) = 6$ and $f(2) = 11$, then the graph of the function $f(x)$ cuts the ordinate $x = 1$ at the point

- (A) (1, 8) (B) (1, -2)
(C) (1, 4) (D) None of these

10. The domain of the function $y = \underbrace{\log_{10} \log_{10} \log_{10} \dots \log_{10} x}_{n \text{ times}}$

- (A) $[10^n, \infty)$ (B) $(10^{n-1}, \infty)$
(C) $(10^{n-2}, \infty)$ (D) None of these

11. Let $f(x) = \sin^2(x/2) + \cos^2(x/2)$ and $g(x) = \sec^2 x - \tan^2 x$. The two functions are equal over the set

- (A) ϕ
(B) \mathbb{R}
(C) $\mathbb{R} - \left\{ x : x = (2n+1)\frac{\pi}{2}, n \in \mathbb{I} \right\}$
(D) None of these

12. If $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cdot \cos\left(x + \frac{\pi}{3}\right)$ and

- $g(5/4) = 1$, then $(g \circ f) x$ is
(A) A polynomial of the first degree in $\sin x, \cos x$
(B) A constant function
(C) A polynomial of the second degree in $\sin x, \cos x$
(D) None of the above

13. If the function $f: [1, \infty) \rightarrow [1, \infty)$ is defined by $f(x) = 2x(x-1)$, then $f^{-1}(x)$ is

- (A) $\left(\frac{1}{2}\right)^{x(x-1)}$ (B) $\frac{1}{2}(1 + \sqrt{1 + 4 \log_2 x})$
(C) $\frac{1}{2}(1 - \sqrt{1 + 4 \log_2 x})$ (D) Not defined

14. The period of $e^{\cos^4 \pi x + x - [x] + \cos^2 \pi x}$ is ($[.]$ denotes the greatest integer function)

- (A) 2 (B) 1
(C) 0 (D) -1

15. Period of the function $f(x) = \frac{\sin\{\sin(nx)\}}{\tan\left(\frac{x}{n}\right)}$, $n \in \mathbb{N}$, is 6π

- then n is equal to
(A) 1 (B) 2
(C) 3 (D) None

16. $f(x) = (\sin x^7) e^{x^5 \sin x^9}$ is

- (A) An even function (B) An odd function
(C) Neither even nor odd (D) None of these

17. Let f be a function satisfying $2f(xy) = \{f(x)\}^y + \{f(y)\}^x$ and

$$f(1) = k \neq 1, \text{ then } \sum_{r=1}^n f(r) \text{ is equal to}$$

- (A) $k^n - 1$ (B) k^n
(C) $k^n + 1$ (D) None

18. The total of solutions of $2^x + 3^x + 4^x - 5^x = 0$ is.
 (A) 0 (B) 1
 (C) 2 (D) None
19. The greatest value of the function $f(x) = \cos \{x e^{\lfloor x \rfloor} + 2x^2 - x\}$, $x \in (-1, \infty)$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x is.
 (A) 0 (B) 1
 (C) 2 (D) 3
20. The graph of $f(x) = \left\lfloor \left(\frac{1}{|x|} - n \right) \right\rfloor - n$ is lie in the ($n > 0$)
 (A) I and II quadrant (B) I and III quadrant
 (C) I and IV quadrant (D) II and III quadrant
21. Let $A \equiv \{1, 2, 3, 4\}$, $B \equiv \{a, b, c\}$, then number of functions from $A \rightarrow B$, which are not onto is.
 (A) 8 (B) 24
 (C) 45 (D) 6
22. Range of the function f defined by $f(x) = \left[\frac{1}{\sin \{x\}} \right]$ (where $\lfloor . \rfloor$ and $\{ . \}$ respectively denotes the greatest integer and the fractional part function) is
 (A) I, the set of natural numbers
 (B) \mathbb{N} , the set of natural numbers
 (C) \mathbb{W} , the set of whole numbers
 (D) \mathbb{Q} , the set of rational numbers
23. If $f(x) = -\frac{x|x|}{1+x^2}$, then $f^{-1}(x)$ equals.
 (A) $\sqrt{\frac{|x|}{1-|x|}}$ (B) $\text{Sgn} \left(\sqrt{\frac{|x|}{1-|x|}} \right)$
 (C) $-\sqrt{\frac{x}{1-x}}$ (D) None of these
24. If $f(x)$ is an even function and satisfies the relation $x^2 f(x) - 2f\left(\frac{1}{x}\right) = g(x)$, where $g(x)$ is an odd function, then the value of $f(5)$ is.
 (A) 0 (B) $\frac{37}{75}$
 (C) 4 (D) $\frac{51}{77}$
25. The domain of the function $f(x) = \log_{3+x}(x^2 - 1)$ is
 (A) $(-3, -1) \cup (1, \infty)$
 (B) $[-3, -1) \cup [1, \infty)$
 (C) $(-3, -2) \cup (-2, -1) \cup (1, \infty)$
 (D) $[-3, -2) \cup (-2, -1) \cup [1, \infty)$
26. The domain of $f(x) = \log|\log x|$ is
 (A) $(0, \infty)$ (B) $(1, \infty)$
 (C) $(0, 1) \cup (1, \infty)$ (D) $(-\infty, 1)$
27. The domain of the function $f(x) = \frac{x}{\sqrt{{}^{10}C_{x-1} - 3 \times {}^{10}C_x}}$, ($n \in \mathbb{Z}$) is
 (A) $(e^{2n\pi}, e^{(3n+1/2)\pi})$ (B) $(e^{(2n+1/4)\pi}, e^{(2n+5/4)\pi})$
 (C) $(e^{2n+1/4}\pi, e^{(3n-3/4)\pi})$ (D) None of these
28. The domain of the following function is
 $f(x) = \log_2 \left(-\log_{1/2} \left(1 + \frac{1}{x^{1/4}} \right) - 1 \right)$
 (A) $(0, 1)$ (B) $(0, 1]$
 (C) $[1, \infty)$ (D) $(1, \infty)$
29. The domain of definition of the function $f(x) = \{x\}^{\lfloor x \rfloor} + \lfloor x \rfloor^{\{x\}}$ is (where $\{ . \}$ represents fractional part and $\lfloor . \rfloor$ represents greatest integral function)
 (A) $\mathbb{R} - \mathbb{I}$ (B) $\mathbb{R} - [0, 1)$
 (C) $\mathbb{R} - \cup \{ \mathbb{I} \cup (0, 1) \}$ (D) $\mathbb{I} \cup (0, 1)$
30. Let $h(x) = |kx + 5|$, the domain of $f(x)$ be $[-5, 7]$, the domain of $f(h(x))$ be $[-6, 1]$, and the range of $h(x)$ be the same as the domain of $f(x)$. Then the value of k is
 (A) 1 (B) 2
 (C) 3 (D) 4
31. The range of the following function is.
 $f(x) = \sqrt{(1 - \cos x)} \sqrt{(1 - \cos x)} \sqrt{(1 - \cos x)} \sqrt{\dots \infty}$
 (A) $[0, 1]$ (B) $[0, 1/2]$
 (C) $[0, 2]$ (D) None
32. The domain of $f(x)$ is $(0, 1)$. Then the domain of $f(e^x) + f(\ln|x|)$ is
 (A) $(-1, e)$ (B) $(1, e)$
 (C) $(-e, -1)$ (D) $(-e, 1)$
33. The domain of $f(x) = \frac{1}{\sqrt{|\cos x| + \cos x}}$ is.
 (A) $[-2n\pi, 2n\pi], n \in \mathbb{Z}$
 (B) $(2n\pi, \overline{2n+1}\pi), n \in \mathbb{Z}$
 (C) $\left(\frac{(4n+1)\pi}{2}, \frac{(4n+3)\pi}{2} \right), n \in \mathbb{Z}$
 (D) $\left(\frac{(4n-1)\pi}{2}, \frac{(4n+1)\pi}{2} \right), n \in \mathbb{Z}$

34. If the graph of the function $f(x) = \frac{a^x - 1}{x^n(a^x + 1)}$ is symmetrical about the y-axis, then n equals
 (A) 2 (B) $\frac{2}{3}$
 (C) $\frac{1}{4}$ (D) $-\frac{1}{3}$
35. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous and differentiable function such that $(f(x^2 + 1))^{y^x} = 5$ for $\forall x \in (0, \infty)$. Then the value of $\left(f\left(\frac{16 + y^2}{y^2}\right)\right)^{\frac{4}{\sqrt{y}}}$ for $y \in (0, \infty)$ is equal to
 (A) 5 (B) 25
 (C) 125 (D) 625
36. Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be defined by $f(x) = x^2 + x + 1, x \in \mathbb{N}$. Then f is
 (A) One-one onto (B) Many-one onto
 (C) One-one but not onto (D) None
37. Let $f(x) = \sqrt{|x| - \{x\}}$ (where $\{.\}$ denotes the fractional part of x and X, Y are its domain and range, respectively). Then
 (A) $x \in \left(-\infty, \frac{1}{2}\right]$ and $Y \in \left[\frac{1}{2}, \infty\right)$
 (B) $x \in \left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ and $Y \in \left[\frac{1}{2}, \infty\right)$
 (C) $X \in \left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ and $Y \in [0, \infty)$
 (D) None
38. The range of $f(x) = [1 + \sin x] + \left[2 + \sin \frac{x}{2}\right] + \left[3 + \sin \frac{x}{3}\right] + \dots + \left[n + \sin \frac{x}{n}\right] \forall x \in [0, \pi]$, where $[.]$ denotes the greatest integer function, is
 (A) $\left\{\frac{n^2 + n - 2}{2}, \frac{n(n+1)}{2}\right\}$
 (B) $\left\{\frac{n(n+1)}{2}\right\}$
 (C) $\left\{\frac{n^2 + n - 2}{2}, \frac{n(n+1)}{2}, \frac{n^2 + n + 2}{2}\right\}$
 (D) $\left\{\frac{n(n+1)}{2}, \frac{n^2 + n + 2}{2}\right\}$
39. If $F(n+1) = \frac{2F(n)+1}{2}, n=1, 2, \dots$ and $F(1)=2$. Then $F(101)$ equals
 (A) 52 (B) 49
 (C) 48 (D) 51
40. The domain of the function $f(x) = \frac{1}{\sqrt{{}^{10}C_{x-1} - 3 \times {}^{10}C_x}}$ contains the points
 (A) 9, 10, 11 (B) 9, 10, 12
 (C) All natural numbers (D) None
41. The domain of the function $f(x) = \sqrt{\log\left(\frac{1}{|\sin x|}\right)}$
 (A) $\mathbb{R} - \{-\pi, \pi\}$ (B) $\mathbb{R} - \{n\pi \mid n \in \mathbb{Z}\}$
 (C) $\mathbb{R} - \{2n\pi \mid n \in \mathbb{Z}\}$ (D) $(-\infty, \infty)$
42. If $f(x) = \text{maximum}\left\{x^3, x^2, \frac{1}{64}\right\} \forall x \in [0, \infty)$, then
 (A) $f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ x^3, & x > 1 \end{cases}$
 (B) $f(x) = \begin{cases} \frac{1}{64}, & 0 \leq x \leq \frac{1}{4} \\ x^2, & \frac{1}{4} < x \leq 1 \\ x^3, & x > 1 \end{cases}$
 (C) $f(x) = \begin{cases} \frac{1}{64}, & 0 \leq x \leq \frac{1}{8} \\ x^2, & \frac{1}{8} < x \leq 1 \\ x^3, & x > 1 \end{cases}$
 (D) $f(x) = \begin{cases} \frac{1}{64}, & 0 \leq x \leq \frac{1}{8} \\ x^3, & x > 1/8 \end{cases}$
43. Let $f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$. Then $f(1) + f(2) + f(3) + \dots + f(n)$ is equal to
 (A) $nf(n) - 1$ (B) $(n+1)f(n) - n$
 (C) $(n+1)f(n) + n$ (D) $nf(n) + n$

44. Let $f(x) = e^{\{e^{|x|} \operatorname{sgn} x\}}$ and $g(x) = e^{\lfloor e^{|x|} \operatorname{sgn} x \rfloor}$, $x \in \mathbb{R}$, where $\{ \}$ and $\lfloor \rfloor$ denote the fractional and integral part functions, respectively. Also, $h(x) = \log(f(x)) + \log(g(x))$. Then for real x , $h(x)$ is
 (A) An odd function
 (B) An even function
 (C) Neither an odd nor an even function
 (D) Both odd and even function

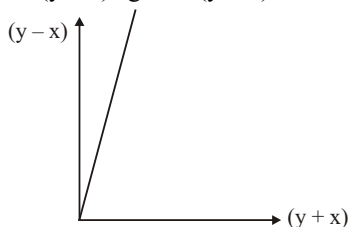
45. The domain of the function $f(x) = \sqrt{x^2 - [x]^2}$, where $[x]$ is the greatest integer less than or equal to x , is
 (A) \mathbb{R}
 (B) $[0, +\infty)$
 (C) $(-\infty, 1]$
 (D) None

46. The period of function $2^{\{x\}} + \sin \pi x + 3^{\{x/2\}} + \cos 2\pi x$ (where $\{x\}$ denotes the fractional part of x) is
 (A) 2
 (B) 1
 (C) 3
 (D) None

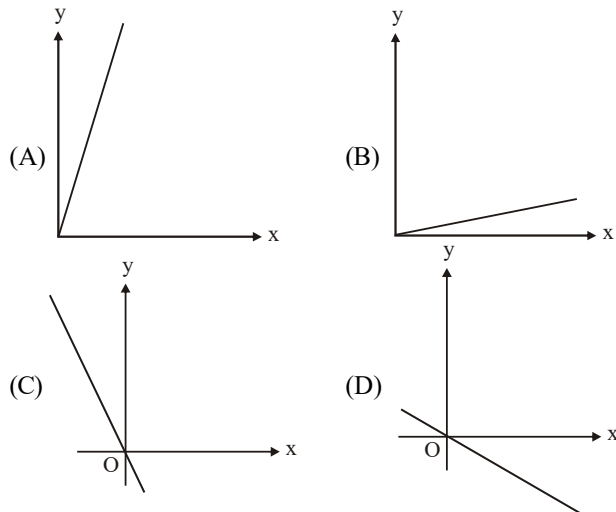
47. The period of the function is
 $f(x) = c \frac{\sin^2 x + \sin^2(x + \frac{\pi}{3}) + \cos x \cos(x + \frac{\pi}{3})}{\sin^2 x + \sin^2(x + \frac{\pi}{3}) + \cos x \cos(x + \frac{\pi}{3})}$ (where c is constant)
 (A) 1
 (B) $\pi/2$
 (C) π
 (D) Cannot be determined

48. $f: \mathbb{N} \rightarrow \mathbb{N}$, where $f(x) = x - (-1)^x$. Then f is
 (A) One-one and into
 (B) Many-one and into
 (C) One-one and onto
 (D) Many-one and onto

49. The graph of $(y - x)$ against $(y + x)$ is shown.



Which one of the following the graph of y against x ?



50. The function f satisfies the functional equation $3f(x) + 2f\left(\frac{x+59}{x-1}\right) = 10x + 30$ for all real $x \neq 1$. The value of $f(7)$ is
 (A) 8
 (B) 4
 (C) -8
 (D) 11

51. If $f(x+y) = f(x) + f(y) - xy - 1 \forall x, y \in \mathbb{R}$ and $f(1) = 1$, then the number of solutions of $f(n) = n$, $n \in \mathbb{N}$, is
 (A) 0
 (B) 1
 (C) 2
 (D) More than 2

52. The domain of the function $f(x) = \frac{1}{\sqrt{\{\sin x\} + \{\sin(\pi + x)\}}}$, where $\{.\}$ denotes the fractional part, is
 (A) $[0, \pi]$
 (B) $(2n+1)\pi/2, n \in \mathbb{Z}$
 (C) $(0, \pi)$
 (D) None

53. Let $f(x) = \begin{cases} \sin x + \cos x, & 0 < x < \frac{\pi}{2} \\ a, & x = \frac{\pi}{2} \\ \tan^2 x + \operatorname{cosec} x, & \frac{\pi}{2} < x < \pi \end{cases}$ Then its odd extension is

(A) $\begin{cases} -\tan^2 x - \operatorname{cosec} x, & -\pi < x < -\frac{\pi}{2} \\ -a, & x = -\frac{\pi}{2} \\ -\sin x + \cos x, & -\frac{\pi}{2} < x < 0 \end{cases}$

(B) $\begin{cases} -\tan^2 x + \operatorname{cosec} x, & -\pi < x < -\frac{\pi}{2} \\ -a, & x = -\frac{\pi}{2} \\ \sin x - \cos x, & -\frac{\pi}{2} < x < 0 \end{cases}$

(C) $\begin{cases} -\tan^2 x + \operatorname{cosec} x, & -\pi < x < -\frac{\pi}{2} \\ a, & x = -\frac{\pi}{2} \\ \sin x - \cos x, & -\frac{\pi}{2} < x < 0 \end{cases}$

(D) $\begin{cases} \tan^2 x + \cos x, & -\pi < x < -\frac{\pi}{2} \\ -a, & x = -\frac{\pi}{2} \\ \sin x + \cos x, & -\frac{\pi}{2} < x < 0 \end{cases}$

54. Let $X = \{a_1, a_2, \dots, a_6\}$ and $Y = \{b_1, b_2, b_3\}$. The number of functions f from x to y such that it is onto and there are exactly three elements x in X such that $f(x) = b_1$ is
 (A) 75 (B) 90
 (C) 100 (D) 120
55. The range of the function $f(x) = \frac{e^x - e^{|x|}}{e^x + e^{|x|}}$ is
 (A) $(-\infty, \infty)$ (B) $[0, 1)$
 (C) $(-1, 0]$ (D) $(-1, 1)$
56. The period of $f(x) = [x] + [2x] + [3x] + [4x] + \dots [nx] - \frac{n(n+1)}{2}x$, where $n \in \mathbb{N}$, is (where $[.]$ represents greatest integer function)
 (A) n (B) 1
 (C) $1/n$ (D) None
57. The function $f : (-\infty, -1) \rightarrow (0, e^5]$ defined by $f(x) = e^{x^3 - 3x + 2}$ is
 (A) Many-one and onto (B) Many-one and into
 (C) One-one and onto (D) One-one and into
58. If the period of $\frac{\cos(\sin(nx))}{\tan\left(\frac{x}{n}\right)}$, $n \in \mathbb{N}$, is 6π , then $n =$
 (A) 3 (B) 2
 (C) 6 (A) 1
59. The values of b and c for which the identity $f(x+1) - f(x) = 8x + 3$ is satisfied, where $f(x) = bx^2 + cx + d$, are
 (A) $b = 2, c = 1$ (B) $b = 4, c = -1$
 (C) $b = -1, c = 4$ (D) $b = -1, c = 1$
60. The domain of the function $f(x) = \left[\log_{10} \left(\frac{5x - x^2}{4} \right) \right]^{1/2}$ is
 (A) $-\infty < x < \infty$ (B) $1 \leq x \leq 4$
 (C) $4 \leq x \leq 16$ (D) $-1 \leq x \leq 1$
61. The number of solutions of the equation $[y + [y]] = 2 \cos x$, where $y = \frac{1}{3} [\sin x + [\sin x + [\sin x]]]$ (where $[.]$ denotes the greatest integer function) is
 (A) 4 (B) 2
 (C) 3 (D) 0

EXERCISE II

1. Which of the following functions are identical ?

- (A) $f(x) = \ln x^2$ and $g(x) = 2 \ln x$
- (B) $f(x) = \log_x e$ and $g(x) = \frac{1}{\log_e x}$
- (C) $f(x) = \sin(\cos^{-1} x)$ and $g(x) = \cos(\sin^{-1} x)$
- (D) None

2. Which of the following functions have the graph symmetrical about the origin ?

- (A) $f(x)$ given by $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$
- (B) $f(x)$ given by $f(x) + f(y) = f\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$
- (C) $f(x)$ given by $f(x+y) = f(x) + f(y) \quad \forall x, y \in \mathbb{R}$
- (D) None

3. If $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a polynomial function satisfying the functional equation $f(f(x)) = 6x - f(x)$, then $f(17)$ is equal to

- (A) 17
- (B) -51
- (C) 34
- (D) -34

4. The domain of the function

$$f(x) = \log_e \left\{ \log_{|\sin x|} (x^2 - 8x + 23) - \frac{3}{\log_2 |\sin x|} \right\}$$

contains which of the following interval(s) ?

- (A) $(3, \pi)$
- (B) $\left(\pi, \frac{3}{2}\right)$
- (C) $\left(\frac{3\pi}{2}, 5\right)$
- (D) None

5. Which of the following function is/are periodic ?

- (A) $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$
- (B) $f(x) = \begin{cases} x - [x]; & 2n \leq x < 2n + 1 \\ \frac{1}{2}; & 2n + 1 \leq x < 2n + 2 \end{cases}$
- (C) $f(x) = (-1)^{\left[\frac{2x}{\pi}\right]}$, where $[.]$ denotes the greatest integer function
- (D) $f(x) = x - [x + 3] + \tan\left(\frac{\pi x}{2}\right)$, where $[.]$ denotes the greatest integer function, and a is a rational number.

6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x+1) = \frac{f(x)-5}{f(x)-3}$

$\forall x \in \mathbb{R}$. Then which of the following statement(s) is/are true ?

- (A) $f(2008) = f(2004)$
- (B) $f(2006) = f(2010)$
- (C) $f(2006) = f(2002)$
- (D) $f(2006) = f(2018)$

7. $f(x) = x^2 - 2ax + a(a+1)$, $f : [a, \infty) \rightarrow [a, \infty)$. If one of the solutions of the equation $f(x) = f^{-1}(x)$ is 5049, then the other may be

- (A) 5051
- (B) 5048
- (C) 5052
- (D) 5050

8. Let $f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & x \geq 3 \end{cases}$ and

$$g(x) = \begin{cases} x - 3, & x < 4 \\ x^2 + 2x + 2, & x \geq 4 \end{cases}$$

Then which of the following is/are true ?

- (A) $(f+g)(3.5) = 0$
- (B) $f(g(3)) = 3$
- (C) $(fg)(2) = 1$
- (D) $(f-g)(4) = 0$

9. If the function f satisfies the relation $f(x+y) + f(x-y) = 2f(x)f(y) \quad \forall x, y \in \mathbb{R}$ and $f(0) \neq 0$, then

- (A) $f(x)$ is an even function
- (B) $f(x)$ is an odd function
- (C) If $f(2) = a$, then $f(-2) = a$
- (D) If $f(4) = b$, then $f(-4) = -b$

10. Consider the function $y = f(x)$ satisfying the condition

$$f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2} \quad (x \neq 0).$$

- (A) Domain of $f(x)$ is \mathbb{R}
- (B) Domain of $f(x)$ is $\mathbb{R} - (-2, 2)$
- (C) Range of $f(x)$ is $[2, \infty)$
- (D) Range of $f(x)$ is $[-2, \infty)$

11. $f : \mathbb{R} \rightarrow [-1, \infty)$ and $f(x) = \ln(|\sin 2x| + |\cos 2x|)$ (where $[.]$ is the greatest integer function). Then,

- (A) $f(x)$ has range \mathbb{Z}
- (B) $f(x)$ is periodic with fundamental period $\pi/4$
- (C) $f(x)$ is invertible in $\left[0, \frac{\pi}{4}\right]$
- (D) $f(x)$ is into function

12. Let $f(x) + f(y) = f(x\sqrt{1-y^2} + y\sqrt{1-x^2})$ [$f(x)$ is not identically zero]. Then

- (A) $f(4x^3 - 3x) + 3f(x) = 0$
- (B) $f(4x^3 - 3x) = 3f(x)$
- (C) $f(2x\sqrt{1-x^2}) + 2f(x) = 0$
- (D) $f(2x\sqrt{1-x^2}) = 2f(x)$

13. If $f(x)$ satisfies the relation $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and $f(1) = 5$, then

- (A) $f(x)$ is an odd function
- (B) $f(x)$ is an even function
- (C) $\sum_{r=1}^m f(r) = 5^{m+1} C_2$
- (D) $\sum_{r=1}^m f(r) = \frac{5m(m+2)}{3}$

14. Consider the real-valued function satisfying $2f(\sin x) + f(\cos x) = x$. Then the

- (A) Domain of $f(x)$ is \mathbb{R}
- (B) Domain of $f(x)$ is $[-1, 1]$
- (C) Range of $f(x)$ is $\left[-\frac{2\pi}{3}, \frac{\pi}{3}\right]$
- (D) Range of $f(x)$ is \mathbb{R}

15. Let $f(x) = \max\{1 + \sin x, 1, 1 - \cos x\}$, $x \in [0, 2\pi]$, and $g(x) = \max\{1, |x-1|\}$, $x \in \mathbb{R}$. Then

- (A) $g(f(0)) = 1$
- (B) $g(f(1)) = 1$
- (C) $f(f(1)) = 1$
- (D) $f(g(0)) = 1 + \sin 1$

16. If $f: \mathbb{R} \rightarrow \mathbb{N} \cup \{0\}$, where $f(\text{area of triangle joining points } P(5, 0), Q(8, 4) \text{ and } R(x, y) \text{ such that angle } PRQ \text{ is a right angle}) = \text{number of triangles}$, then which of the following is true ?

- (A) $f(5) = 4$
- (B) $f(7) = 0$
- (C) $f(6.25) = 2$
- (D) $f(x)$ is into

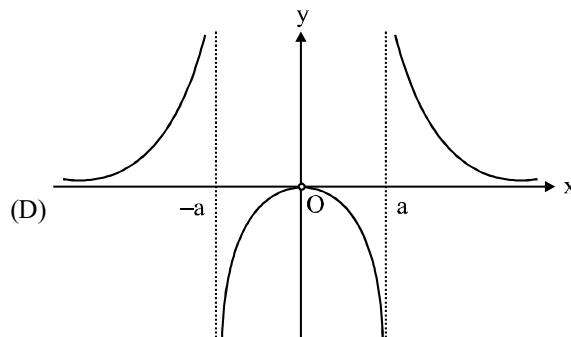
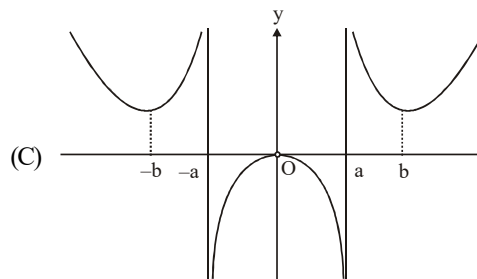
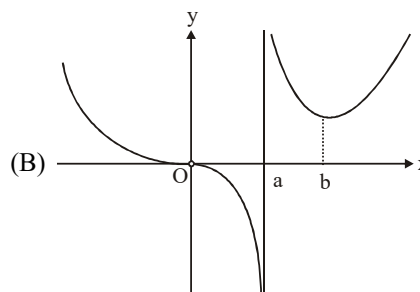
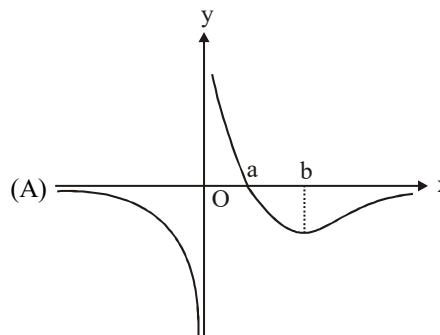
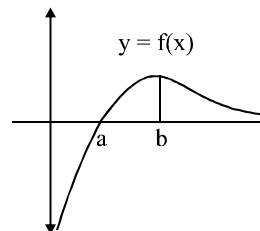
17. Which of the following functions are homogeneous ?

- (A) $x \sin y + y \sin x$
- (B) $x e^{y/x} + y^{x/y}$
- (C) $x^2 - xy$
- (D) $\arcsin xy$

18. $f(x)$ and $g(x)$ are two functions defined for all real values of x . $f(x)$ is an even function and $g(x)$ is periodic function, then

- (A) $f[g(x)]$ is a periodic function
- (B) $g[f(x)]$ is a periodic function
- (C) $f[g(x)]$ is an even function
- (D) $g[f(x)]$ is an even function

19. The graph of function $f(x)$ is as shown, adjacently. Then the graph of $\frac{1}{f(|x|)}$ is



20. If $f(x)$ is defined on $(0, 1)$ then the domain of definition of $f(e^x) + f(\ln|x|)$ is
 (A) $(-e, -1)$ (B) $(-e, -1) \cup (1, e)$
 (C) $(-\infty, 1) \cup (1, \infty)$ (D) $(-e, e)$

21. Given the function $f(x)$ such that $2f(x) +$

$$x f\left(\frac{1}{x}\right) - 2f\left(\sqrt{2} \sin \pi\left(x + \frac{1}{4}\right)\right) = 4 \cos^2 \frac{\pi x}{2} + x \cos \frac{\pi}{x},$$

then which one of the following is correct ?

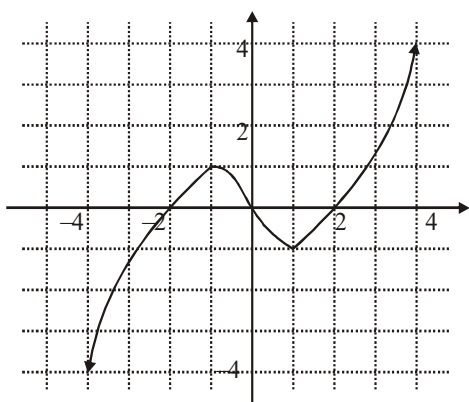
- (A) $f(2) + f(1/2) = 1$
 (B) $f(1) = -1$, but the values of $f(2)$, $f(1/2)$ cannot be determined
 (C) $f(2) + f(1) = f(1/2)$
 (D) $f(2) + f(1) = 0$

22. The function $\cot(\sin x)$ -

- (A) is not defined for $x = (4n + 1) \frac{\pi}{2}$
 (B) is not defined for $x = n\pi$
 (C) lies between $-\cot 1$ and $\cot 1$
 (D) can't lie between $-\cot 1$ and $\cot 1$

23. The graph of $\phi(x)$ is given then the number of positive solution of $|\phi(x) - 1| = 1$ are

- (A) 5 (B) 2
 (C) 3 (D) 1



24. Which of the following function(s) is/are periodic ?

- (A) $f(x) = 3x - [3x]$
 (B) $g(x) = \sin(1/x)$, $x \neq 0$ & $g(0) = 0$
 (C) $h(x) = x \cos x$
 (D) $w(x) = \sin(\sin(\sin x))$

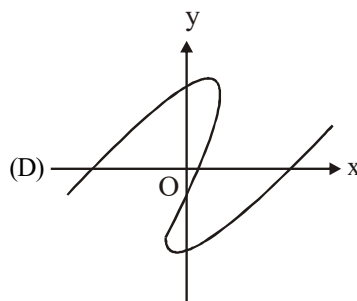
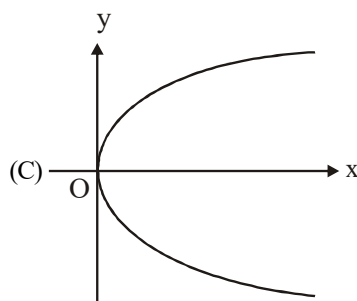
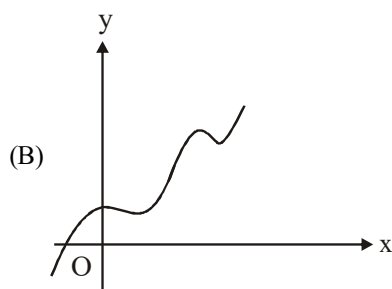
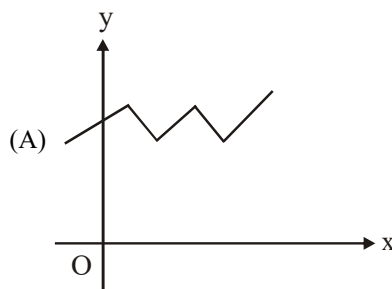
25. Which of the following functions are not homogeneous ?

- (A) $x + y \cos \frac{y}{x}$ (B) $\frac{xy}{x + y^2}$
 (C) $\frac{x - y \cos x}{y \sin x + y}$ (D) $\frac{x}{y} \ln\left(\frac{y}{x}\right) + \frac{y}{x} \ln\left(\frac{x}{y}\right)$

26. Which of the functions defined below are NOT one-one function(s) ?

- (A) $f(x) = 5(x^2 + 4)$, $(x \in \mathbb{R})$ (B) $g(x) = 2x + (1/x)$
 (C) $h(x) = \ln(x^2 + x + 1)$, $(x \in \mathbb{R})$ (D) $f(x) = e^{-x}$

27. Which of the following graphs are graphs of functions.



28. If domain of f is D_1 and domain of g is D_2 , then domain of $f + g$ is

- (A) $D_1 \setminus D_2$ (B) $D_1 - (D_1 \setminus D_2)$
 (C) $D_2 - (D_2 \setminus D_1)$ (D) $D_1 \cap D_2$

29. Let $f(x) = 2x - \sin x$ and $g(x) = \sqrt[3]{x}$, then

- (A) Range of $g \circ f$ is \mathbb{R} (B) $g \circ f$ is one-one
 (C) both f and g are one-one (D) both f and g are onto

30. Let $f(x) = \begin{cases} 0, & \text{for } x = 0 \\ x^2 \sin\left(\frac{\pi}{x}\right), & \text{for } -1 < x < 1, (x \neq 0), \text{ then} \\ x|x|, & \text{for } x \geq 1 \text{ or } x \leq -1 \end{cases}$

- (A) $f(x)$ is an odd function
- (B) $f(x)$ is an even function
- (C) $f(x)$ is neither odd nor even
- (D) $f'(x)$ is an even function

31. If $e^x + e^{f(x)} = e$, then for $f(x)$

- (A) domain = $(-\infty, 1)$
- (B) range = $(-\infty, 1)$
- (C) domain = $(-\infty, 0]$
- (D) range = $(-\infty, 1]$

32. Let $f(x) = \frac{5\sqrt{\sin 2x}}{1 + \sqrt[3]{\sin x}}$. If D is the domain of f , then D contains

- (A) $(0, \pi)$
- (B) $(-2\pi, -\pi)$
- (C) $(2\pi, 3\pi)$
- (D) $(4\pi, 6\pi)$

33. If $y = f(x) = \frac{x+2}{x-1}$, then

- (A) $x = f(y)$
- (B) $f(1) = 3$
- (C) y increases with x for $x < 1$
- (D) f is rational function of x

34. If $[x]$ denotes the greatest integer less than or equal to x , the extreme values of the function $f(x) = [1 + \sin x] + [1 + \sin 2x] + [1 + \sin 3x] + \dots + [1 + \sin nx]$, $n \in \mathbb{I}^+$, $x \in (0, \pi)$ are

- (A) $n - 1$
- (B) n
- (C) $n + 1$
- (D) $n + 2$

35. If $f(x) = \cos([\pi^2]x) + \cos([-\pi^2]x)$, where $[x]$ stands for the greatest integer function, then

- (A) $f\left(\frac{\pi}{2}\right) = -1$
- (B) $f(\pi) = 1$
- (C) $f(-\pi) = 0$
- (D) $f\left(\frac{\pi}{4}\right) = 1$

36. $f(x) = \cos^2 x + \cos^2\left(\frac{\pi}{3} + x\right) - \cos x \cos\left(\frac{\pi}{3} + x\right)$ is

- (A) An odd function
- (B) An even function
- (C) A periodic function
- (D) $f(0) = f(1)$

37. Which of the following functions are not identical ?

(A) $f(x) = \frac{x}{x^2}$ and $g(x) = \frac{1}{x}$

(B) $f(x) = \frac{x^2}{x}$ and $g(x) = x$

(C) $f(x) = \ln x^4$ and $g(x) = 4 \ln x$

(D) $f(x) = \ln \{(x-1)(x-2)\}$ and $g(x) = \ln(x-2) + \ln(x-3)$

38. The possible values of 'a' for which the function $f(x) = e^{x-[x]} + \cos ax$ (where $[.]$ denotes the greatest integer function) is periodic with finite fundamental period is

- (A) π
- (B) 2π
- (C) 3π
- (D) 1

39. Let f be the greatest integer function and g be the modulus functions, then

(A) $(g \circ f - f \circ g)\left(-\frac{5}{3}\right) = 1$

(B) $(f+2g)(-1) = 1$

(C) $(g \circ f - f \circ g)\left(\frac{5}{3}\right) = 0$

(D) $(f+2g)(1) = 1$

40. Which of the following function are even ?

(A) $f(x) = x \left(\frac{a^x + 1}{a^x - 1} \right)$

(B) $g(x) = \ln(x + \sqrt{x^2 + a^2})$

(C) $h(x) = \sqrt[3]{(1-x)^2} + \sqrt[3]{(1-x)^2}$

(D) $p(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

41. Which of the following functions are periodic ?

(A) $f(x) = \sin x + |\sin x|$

(B) $g(x) = \frac{(1 + \sin x)(1 + \sec x)}{(1 + \cos x)(1 + \operatorname{cosec} x)}$

(C) $h(x) = \max(\sin x, \cos x)$

(D) $p(x) = [x] + \left[x + \frac{1}{3} \right] \left[x + \frac{2}{3} \right] - 3x + 10$, where

$[.]$ denotes the greatest integer function.

42. If $f(x) = \frac{x}{x^2 + 1}$ and $f(A) = \left\{ y : -\frac{1}{2} \leq y < 0 \right\}$, then set A is

- (A) $[-1, 0)$
- (B) $(-\infty, -1)$
- (C) $(-\infty, 0)$
- (D) $(-\infty, \infty)$

EXERCISE III

[PART 1 - COMPREHENSION TYPE]

Comprehension - 1

The accompanying figure shows the graph of a function $f(x)$ with domain $[-3, 4]$ and range $[-1, 2]$

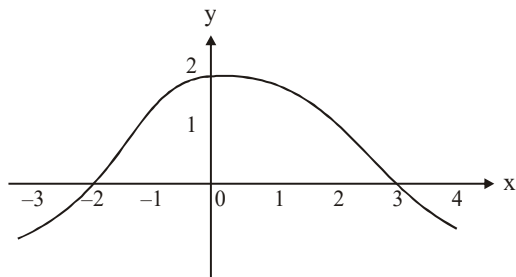


Figure (i)

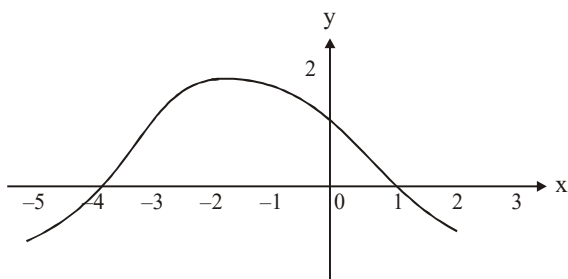


Figure (ii)

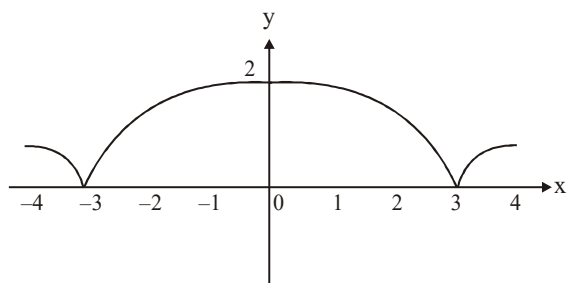


Figure (iii)

On the basis of above information, answer the following questions :

- Figure (ii) represents the graph of the function
 (A) $f(x)$ (B) $f(x-2)$
 (C) $f(x+2)$ (D) $f(x-1)+1$
- Figure (iii) represents the graph of the function
 (A) $f(x)$ (B) $f(|x|)$
 (C) $|f(x)|$ (D) $|f(|x|)|$
- The domain and range respectively of
 (A) $f(-x)$ are $[-4, 3]$ and $[-2, 1]$
 (B) $f(x)-1$ are $[-3, 4]$ and $[-1, 2]$
 (C) $f(x)+2$ are $[-3, 4]$ and $[-2, 4]$
 (D) $-f(x+1)+1$ are $[-4, 3]$ and $[-1, 2]$

- $[-2, 5]$ and $[-2, 1]$ are the domain and range respectively of the function
 (A) $-f(x)$ (B) $f(x-1)$
 (C) $-f(x+1)+1$ (D) $-f(x+1)$
- The number of solutions of figure (iii) and $(2x-6)^2+4y^2=49$ are
 (A) 2 (B) 4
 (C) 6 (D) None

Comprehension - 2

Let f be a function satisfying

$$f(x) = \frac{a^x}{a^x + \sqrt{a}} = g_a(x) \quad (a > 0)$$

- Let $f(x) = g_3(x)$, then the value of $\left[\sum_{r=1}^{1995} f\left(\frac{r}{1996}\right) \right]$ is (where $[.]$ denotes the greatest integer function)
 (A) 995 (B) 996
 (C) 997 (D) 998
- Let $f(x) = g_4(x)$, then $\sum_{r=1}^{1996} f\left(\frac{r}{1997}\right)$ is
 (A) Zero (B) Even
 (C) Odd (D) None
- The value of $g_3(x) + g_3(1-x)$ is
 (A) 1 (B) 5
 (C) 10 (D) None
- The value of $\sum_{r=1}^{2n-1} 2f\left(\frac{r}{2n}\right)$ is
 (A) 0 (B) $2n-1$
 (C) $2n$ (D) None
- If the value of $\sum_{r=0}^{2n} 2f\left(\frac{r}{2n+1}\right) = \frac{1}{1+\sqrt{a}} + 987$, then the value of n is
 (A) 493 (B) 494
 (C) 987 (D) 988

Comprehension - 3

Let $f(x) = x^2 - 5x + 6$, $g(x) = f(|x|)$, $h(x) = |g(x)|$ and $\phi(x) = h(x) - (x)$ are four functions, where (x) is the least integral function of $x \geq x$.

On the basis of above information, answer the following questions :

- The number of solutions of the equation $g(x) = 0$ is
 (A) 0 (B) 2
 (C) 4 (D) 6
- The value of λ for which the equation $g(x) - \lambda = 0$ has exactly three real and distinct roots
 (A) 2 (C) 4
 (C) 6 (D) None

13. The set of values of μ such that the equation $h(x) - \mu = 0$ has exactly eight real and distinct roots

- (A) $\mu \in \left(0, \frac{1}{2}\right)$ (B) $\mu \in \left(0, \frac{1}{4}\right)$
 (C) $\mu \in \left(0, \frac{1}{2}\right]$ (D) $\mu \in \left(0, \frac{1}{4}\right]$

14. The set of all values of x , such that equation $g(x) + |g(x)| = 0$ is satisfied

- (A) $[-3, -2]$ (B) $[2, 3]$
 (C) $[-3, -2] \cup [2, 3]$ (D) ϕ

15. Which statement is correct for $\phi(x) = 0$

- (A) One value of x is satisfied for $\phi(x) = 0$ and that x lie between 4 and 5
 (B) One value of x is satisfied for $\phi(x) = 0$ and that x lie between 3 and 4
 (C) Two values of x is satisfied for $\phi(x) = 0$
 (D) None

Comprehension - 4

Let $f(x) = \min \{x - [x], -x - [-x]\}$, $-2 \leq x \leq 2$; $g(x) = |2 - |x - 2||$, $-2 \leq x \leq 2$ and $h(x) = \frac{|\sin x|}{\sin x}$, $-2 \leq x \leq 2$ and $x \neq 0$ (where $[x]$ denotes the greatest integer function $\leq x$)

16. The number of solutions of the equation $x^2 + [f(x)]^2 = 1$ is $\{-1 \leq x \leq 1\}$

- (A) 0 (B) 2
 (C) 4 (D) 6

17. The range of $f(x)$ is

- (A) $\left[0, \frac{1}{2}\right]$ (B) $[0, 1]$
 (C) $[0, 2]$ (D) None

18. The sum of all the roots of the equation $g(x) - h(x) = 0$ is $\{-2 \leq x \leq 2\}$

- (A) Positive (B) Negative
 (C) Zero (D) None

19. The set of values of α such that the equation $f(x) - \alpha = 0$ has exactly eight real and distinct roots

- (A) $\alpha \in \left(0, \frac{1}{2}\right)$ (B) $\alpha \in \left[0, \frac{1}{2}\right)$
 (C) $\alpha \in [0, 1)$ (D) $\alpha \in (0, 1)$

20. The value of $\int_{-2}^2 f(x) dx$ is

- (A) 0 (B) 1
 (C) 2 (D) 8

Comprehension - 5

Let $F(x) = f(x) + g(x)$, $G(x) = f(x) - g(x)$ and $H(x) = \frac{f(x)}{g(x)}$, where $f(x) = 1 - 2\sin^2 x$ and $g(x) = 2\cos x$, $\forall f: \mathbb{R} \rightarrow [-1, 1]$ and $g: \mathbb{R} \rightarrow [-1, 1]$.

21. Domain and range of $H(x)$ are respectively

- (A) \mathbb{R} and $\{1\}$
 (B) \mathbb{R} and $\{0, 1\}$
 (C) $\mathbb{R} \sim \{(2n+1)\frac{\pi}{4}\}$, and $\{1\}$, $n \in \mathbb{I}$
 (D) $\mathbb{R} \sim \left\{(2n+1)\frac{\pi}{2}\right\}$, and $\{0, 1\}$, $n \in \mathbb{I}$

22. If $F: \mathbb{R} \rightarrow [-2, 2]$, then

- (A) $F(x)$ is one-one function (B) $F(x)$ is onto function
 (C) $F(x)$ is into function (D) none of these

23. Which statement is correct ?

- (A) period of $f(x)$, $g(x)$ and $F(x)$ makes are AP with common difference $\pi/3$
 (B) period of $f(x)$, $g(x)$ and $F(x)$ are same and is equal to 2π
 (C) sum of period of $f(x)$, $g(x)$ and $F(x)$ is 3π
 (D) sum of period of $f(x)$, $g(x)$ and $F(x)$ is 6π

24. Which statement is correct

- (A) the domain of $G(x)$ and $H(x)$ are same
 (B) the range of $G(x)$ and $H(x)$ are same
 (C) the union of domain of $G(x)$ and $H(x)$ are all real numbers
 (D) the union of domain of $G(x)$ and $H(x)$ are rational numbers

25. If the solution of $F(x) - G(x) = 0$ are $x_1, x_2, x_3, \dots, x_n$ where $x \in [0, 5\pi]$, then

- (A) $x_1, x_2, x_3, \dots, x_n$ are in AP with common difference $\pi/4$
 (B) the number of solutions of $F(x) - G(x) = 0$ is 10, $\forall x \in [0, 5\pi]$
 (C) the sum of all solutions of $F(x) - G(x) = 0$, $\forall x \in [0, 5\pi]$ is π
 (D) (B) and (C) are correct

Comprehension - 6

Consider the functions

$$f(x) = \begin{cases} x+1, & x \leq 1 \\ 2x+1 & 1 < x \leq 2 \end{cases} \text{ and } g(x) = \begin{cases} x^2, & -1 \leq x < 2 \\ x+2, & 2 \leq x \leq 3 \end{cases}$$

26. The domain of the function $f(g(x))$ is

- (A) $[0, \sqrt{2}]$ (B) $[-1, 2]$
 (C) $[-1, \sqrt{2}]$ (D) None

27. The range of the function $f(g(x))$ is
 (A) $[1, 5]$ (B) $[2, 3]$
 (C) $[1, 2] \cup (3, 5]$ (D) None
28. The number of roots of the equation $f(g(x)) = 2$ is
 (A) 1 (B) 2
 (C) 4 (D) None

Comprehension - 7

Consider the functions

$$f(x) = \begin{cases} [x], & -2 \leq x \leq -1 \\ |x| + 1, & -1 < x \leq 2 \end{cases} \quad \text{and}$$

$$g(x) = \begin{cases} [x], & -\pi \leq x < 0 \\ \sin x, & 0 \leq x \leq \pi \end{cases}$$

where $[.]$ denotes the greatest integer function.

29. The exhaustive domain of $g(f(x))$ is
 (A) $[0, 2]$ (B) $[-2, 0]$
 (C) $[-2, 2]$ (D) $[-1, 2]$
30. The range of $g(f(x))$ is
 (A) $[\sin 3, \sin 1]$ (B) $[\sin 3, 1] \cup \{-2, -1, 0\}$
 (C) $[\sin 1, 1] \cup \{-2, -1\}$ (D) $[\sin 1, 1]$
31. The number of integral points in the range of $g(f(x))$ is
 (A) 2 (B) 4
 (C) 3 (D) 5

Comprehension - 8

Let $f(x) = f_1(x) - 2f_2(x)$, where

$$f_1(x) = \begin{cases} \min\{x^2, |x|\}, & |x| \leq 1 \\ \max\{x^2, |x|\}, & |x| > 1 \end{cases} \quad \text{and}$$

$$f_2(x) = \begin{cases} \min\{x^2, |x|\}, & |x| > 1 \\ \max\{x^2, |x|\}, & |x| \leq 1 \end{cases}$$

and let $g(x) = \begin{cases} \min\{f(t) : -3 \leq t \leq x, & -3 \leq x < 0\} \\ \max\{f(t) : 0 \leq t \leq x, & 0 \leq x \leq 3\} \end{cases}$

32. For $-3 \leq x \leq -1$, the range of $g(x)$ is
 (A) $[-1, 3]$ (B) $[-1, -15]$
 (C) $[-1, 9]$ (D) None
33. For $x \in (-1, 0)$, $f(x) + g(x)$ is
 (A) $x^2 - 2x + 1$ (B) $x^2 + 2x - 1$
 (C) $x^2 + 2x + 1$ (D) $x^2 - 2x - 1$
34. The graph of $y = g(x)$ in its domain is broken at
 (A) 1 point (B) 2 points
 (C) 3 points (D) None

Comprehension - 9

If $a_0 = x$, $a_{n+1} = f(a_n)$, where $n = 0, 1, 2, \dots$, then answer the following questions.

35. If $f(x) = \sqrt[m]{a - x^m}$, $x > 0$, $m \geq 2$, $m \in \mathbb{N}$, then
 (A) $a_n = x$, $n = 2k + 1$, where k is an integer
 (B) $a_n = f(x)$ if $n = 2k$, where k is an integer
 (C) The inverse of a_n exists for any real value of n and m
 (D) None
36. If $f(x) = \frac{1}{1-x}$, then which of the following is not true ?
 (A) $a_n = \frac{1}{1-x}$ if $n = 3k + 1$ (B) $a_n = \frac{x-1}{x}$ if $n = 3k + 2$
 (C) $a_n = x$ if $n = 3k$ (D) None
37. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = 3 + 4x$ and $a_n = A + Bx$, then which of the following is not true ?
 (A) $A + B + 1 = 2^{2n+1}$ (B) $|A - B| = 1$
 (C) $\lim_{n \rightarrow \infty} \frac{A}{B} = -1$ (D) None

Comprehension - 10

Let $f: \mathbb{N} \rightarrow \mathbb{R}$ be a function satisfying the following conditions :

$$f(1) = 1/2 \text{ and } f(1) + 2f(2) + 3f(3) + \dots + nf(n) = n(n+1),$$

$$f(n) \text{ for } n \geq 2.$$

38. The value of $f(1003)$ is $\frac{1}{K}$, where K equals
 (A) 1003 (B) 2003
 (C) 2005 (D) 2006
39. The value of $f(999)$ is $\frac{1}{K}$, where K equals
 (A) 999 (B) 1000
 (C) 1998 (D) 2000
40. $f(1), f(2), f(3), f(4), \dots$ represent a series of
 (A) an HP (B) a GP
 (C) an HP (D) an arithmatico-geometric progression

Comprehension - 11

Consider the function $f(x)$ satisfying the identity $f(x) +$

$$f\left(\frac{x-1}{x}\right) = 1 + x \quad \forall x \in \mathbb{R} - \{0, 1\}, \text{ and } g(x) = 2f(x) - x + 1.$$

41. The domain of $y = \sqrt{g(x)}$ is
 (A) $\left(-\infty, \frac{1-\sqrt{5}}{2}\right] \cup \left[1, \frac{1+\sqrt{5}}{2}\right)$

(B) $\left(-\infty, \frac{1-\sqrt{5}}{2}\right] \cup (0, 1) \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$

(C) $\left[\frac{-1-\sqrt{5}}{2}, 0\right] \cup \left[\frac{-1+\sqrt{5}}{2}, 1\right]$

(D) None

42. The range of $y = g(x)$ is

(A) $(-\infty, 5]$ (B) $[1, \infty)$

(C) $(\infty, 1] \cup [5, \infty)$ (D) None

43. The number of roots of the equation $g(x) = 1$ is

(A) 2 (B) 1

(C) 3 (D) 0

Comprehension - 12

If $(f(x))^2 \times f\left(\frac{1-x}{1+x}\right) = 64x \quad \forall x \in Df$, then

44. $f(x)$ is equal to

(A) $4x^{2/3} \left(\frac{1+x}{1-x}\right)^{1/3}$ (B) $x^{1/3} \left(\frac{1+x}{1-x}\right)^{1/3}$

(C) $x^{1/3} \left(\frac{1-x}{1+x}\right)^{1/3}$ (D) $x \left(\frac{1+x}{1-x}\right)^{1/3}$

45. The domain of $f(x)$ is

(A) $[0, \infty)$ (B) $\mathbb{R} - \{1\}$

(C) $(-\infty, \infty)$ (D) None

46. The value of $f(9/7)$ is

(A) $8(7/9)^{2/3}$ (B) $4(9/7)^{1/3}$

(C) $-8(9/7)^{2/3}$ (D) None

Comprehension - 13

$$\text{Let } f(x) = \begin{cases} 2x + a, & x \geq -1 \\ bx^2 + 3, & x < -1 \end{cases} \text{ and } g(x) = \begin{cases} x + 4, & 0 \leq x \leq 4 \\ -3x - 2, & -2 < x < 0 \end{cases}$$

47. $g(f(x))$ is not defined if

(A) $a \in (10, \infty), b \in (5, \infty)$

(B) $a \in (4, 10), b \in (5, \infty)$

(C) $a \in (10, \infty), b \in (0, 1)$

(D) $a \in (4, 10), b \in (1, 5)$

48. If domain of $g(f(x))$ is $[-1, 4]$, then

(A) $a = 1, b > 5$ (B) $a = 2, b > 7$

(C) $a = 2, b > 10$ (D) $a = 0, b \in \mathbb{R}$

49. If $a = 2$ and $b = 3$, then the range of $g(f(x))$ is

(A) $(-2, 8]$

(B) $(0, 8]$

(C) $[4, 8]$

(D) $[-1, 8]$

Comprehension - 14

$$f(x) = \begin{cases} x - 1, & -1 \leq x \leq 0 \\ x^2, & 0 \leq x \leq 1 \end{cases} \text{ and } g(x) = \sin x$$

Consider the functions $h_1(x) = f(g(x))$ and $h_2(x) = |f(g(x))|$.

50. Which of the following is not true about $h_1(x)$?

(A) It is a periodic function with period π .

(B) The range is $[0, 1]$

(C) The domain is \mathbb{R}

(D) None

51. Which of the following is not true about $h_2(x)$?

(A) The domain is \mathbb{R}

(B) It is periodic with period 2π .

(C) The range is $[0, 1]$

(D) None

52. If for $h_1(x)$ and $h_2(x)$ are identical functions, then which of the following is not true ?

(A) Domain of $h_1(x)$ and $h_2(x)$ is $x \in [2n\pi(2n+1)\pi], n \in \mathbb{Z}$

(B) Range of $h_1(x)$ and $h_2(x)$ is $[0, 1]$

(C) Period of $h_1(x)$ and $h_2(x)$ is π .

(D) None of these

Comprehension - 15

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(2-x) = f(2+x)$ and $f(20-x) = f(x) \quad \forall x \in \mathbb{R}$. For this function f , answer the following.

53. If $f(0) = 5$, then the minimum possible number of values of x satisfying $f(x) = 5$, for $x \in [0, 170]$, is

(A) 21

(B) 12

(C) 11

(D) 22

54. The graph of $y = f(x)$ is not symmetrical about

(A) Symmetrical about $x = 2$

(B) Symmetrical about $x = 10$

(C) Symmetrical about $x = 8$

(D) None

55. If $f(2) \neq f(6)$, then the

(A) Fundamental period of $f(x)$ is 1

(B) Fundamental period of $f(x)$ may be 1

(C) Period of $f(x)$ cannot be 1

(D) Fundamental period of $f(x)$ is 8

[PART 2 - ASSERTION AND REASON]

Each question has four choices a, b, c and d out of which only one is correct. Each question contains STATEMENT 1 and STATEMENT 2.

- a. If both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.
- b. If both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.
- c. If STATEMENT 1 is TRUE and STATEMENT 2 is FALSE
- d. If STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1** : If $g(x) = f(x) - 1$, $f(x) + f(1-x) = 2 \forall x \in \mathbb{R}$, then $g(x)$ is symmetrical about the point $(1/2, 0)$

Statement 2 : If $g(a-x) = -g(a+x) \forall x \in \mathbb{R}$, then $g(x)$ is symmetrical about the point $(a, 0)$.

2. **Statement 1** : If $f(x) = \cos x$ and $g(x) = x^2$, then $f(g(x))$ is an even function.

Statement 2 : If $f(g(x))$ is an even function, then both $f(x)$ and $g(x)$ must be even functions.

3. **Statement 1** : $f(x) = \sqrt{ax^2 + bx + c}$ has range $[0, \infty)$ if $b^2 - 4ac > 0$.

Statement 2 : $ax^2 + bx + c = 0$ has real roots if $b^2 - 4ac = 0$,

4. **Statement 1** : $f(x) = \log_e x$ cannot be expressed as the sum of odd and even function.

Statement 2 : $f(x) = \log_e x$ is neither odd nor even function.

5. Consider the function satisfying the relation if

$$f\left(\frac{2 \tan x}{1 + \tan^2 x}\right) = \frac{(1 + \cos 2x)(\sec^2 x + 2 \tan x)}{2}$$

Statement 1 : The range of $y = f(x)$ is \mathbb{R}

Statement 2 : Linear function has range \mathbb{R} if domain is \mathbb{R} .

6. **Statement 1** : $f(x) = \cos(x^2 - \tan x)$ is a non-periodic function.

Statement 2 : $x^2 - \tan x$ is a non-periodic function.

7. **Statement 1** : For a continuous surjective function $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x)$ can never be a periodic function.

Statement 2 : For a surjective function $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x)$ to be periodic, it should necessarily be a discontinuous function.

8. **Statement 1** : The graph of $y = \sec^2 x$ is symmetrical about the y-axis.

Statement 2 : The graph of $y = \tan x$ is symmetrical about the origin.

9. Consider the function $f(x) = \sin(kx) + \{x\}$, where $\{x\}$ represents the fractional part function.

Statement 1 : $f(x)$ is periodic for $k = m\pi$, where m is a rational number.

Statement 2 : The sum of two periodic functions is always periodic.

10. **Statement 1** : $f(x) = \sin x$ and $g(x) = \cos x$ are identical functions.

Statement 2 : Both the functions have the same domain and range.

11. **Statement 1** : $f: \mathbb{N} \rightarrow \mathbb{R}$, $f(x) = \sin x$ is a one-one function.

Statement 2 : The period of $\sin x$ is 2π and 2π is an irrational number.

12. **Statement 1** : The period of $f(x) = \sin x$ is $2\pi \Rightarrow$ the period of $g(x) = |\sin x|$ is π .

Statement 2 : The period of $f(x) = \cos x$ is $2\pi \Rightarrow$ the period of $g(x) = |\cos x|$ is π .

13. **Statement 1** : The solution of equation $\|x^2 - 5x + 4| - |2x - 3| = |x^2 - 3x + 1|$ is $x \in (-\infty, 1] \cup \left[\frac{3}{2}, 4\right]$.

Statement 2 : If $|x + y| = |x| + |y|$, then $x \cdot y \geq 0$

14. Consider the functions $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$, and $g: \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = 3x + 4$.

Statement 1 : $f(g(x))$ is an onto function.

Statement 2 : $g(x)$ is an onto function.

15. Consider the function $f(x) = \log_e x$ and $g(x) = 2x + 3$.

Statement 1 : $f(g(x))$ is a one-one function.

Statement 2 : $g(x)$ is a one-one function.

16. **Statement 1** : The period of the function $f(x) = \sin \{x\}$ is 1, where $\{.\}$ represents fractional part function.

Statement 2 : $g(x) = \{x\}$ has period 1.

17. **Statement 1** : If $f: \mathbb{R} \rightarrow \mathbb{R}$, $y = f(x)$ is a periodic and continuous function, then $y = f(x)$ cannot be onto.

Statement 2 : A continuous periodic function is bounded.

18. **Statement 1** : The period of $f(x) = \sin 3x \cos [3x] - \cos 3x \sin [3x]$ is $\frac{1}{3}$ where $[.]$ denotes the greatest integer function $\leq x$.

Statement 2 : The period of $\{x\}$ is 1, where $\{x\}$ denotes the fractional part function of x .

- (A) A (B) B (C) C (D) D

19. **Statement 1** : The functions $f(x) = x^2 - x + 1$, $x \geq \frac{1}{2}$ and

$$g(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$$

then the number of solutions of the equation $f(x) = g(x)$ is two.

Statement 2 : $f(x)$ and $g(x)$ are mutually inversion.

- (A) A (B) B (C) C (D) D

20. **Statement 1** : If $f(x) = \ln x^2$ and $g(x) = 2 \ln x$, then $f(x) = g(x)$.
Statement 2 : For $x < 0$, $g(x)$ is not defined.
 (A) A (B) B (C) C (D) D
21. **Statement 1** : A function $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x - [x]$ (where $[x]$ is greatest integer $\leq x$) for all $x \in \mathbb{R}$. f is not invertible.
Statement 2 : $f(x)$ is periodic function.
 (A) A (B) B (C) C (D) D
22. **Statement 1** : The domain of $f(x) = \sqrt{\cos(\sin x)}$ and $g(x) = \sqrt{\sin(\cos x)}$ are same.
Statement 2 : $\because -1 \leq \cos(\sin x) \leq 1$ and $-1 \leq \sin(\cos x) \leq 1$
 (A) A (B) B (C) C (D) D
23. **Statement 1** : Every even function $y = f(x)$ are not one-one $\forall x \in D_f$
Statement 2 : Even function is symmetrical about the y-axis.
 (A) A (B) B (C) C (D) D
24. **Statement 1** : $f(x) = \sin x + \cos ax$ is a periodic function.
Statement 2 : a is a rational number
 (A) A (B) B (C) C (D) D
25. **Statement 1** : If $f(x) = x^5 - 16x + 2$, then $f(x) = 0$ has only one root in the interval $[-1, 1]$.
Statement 2 : $f(-1)$ and $f(1)$ are of opposite sign.
 (A) A (B) B (C) C (D) D
26. **Statement 1** : The function $f(x) = |x|$ is not one-one.
Statement 2 : The negative real number are not the images of any real numbers.
 (A) A (B) B (C) C (D) D
27. **Statement 1** : The equation $x^4 = (\lambda x - 1)^2$ has atmost two real solutions (is $\lambda > 0$)
Statement 2 : Curves $f(x) = x^4$ and $g(x) = (\lambda x - 1)^2$ cut atmost two points.
 (A) A (B) B (C) C (D) D

28. **Statement 1** : If $f(x)$ is odd function and $g(x)$ is even function, then $f(x) + g(x)$ is neither even nor odd.
Statement 2 : Odd function is symmetrical in opposite quadrants and even function is symmetrical about the y-axis.
 (A) A (B) B (C) C (D) D
29. **Statement 1** : The least period of the function.
 $f(x) = \cos(\cos x) + \cos(\sin x) + \sin 4x$ is π .
Statement 2 : $\because f(x + \pi) = f(x)$.
 (A) A (B) B (C) C (D) D

[PART 3 - MATCH THE COLUMN]
 TYPE 1

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE OR MORE** statement(s) in **Column-II**.

- | 1. Column-I | Column-II |
|---|--------------|
| (A) $f: \mathbb{R} \rightarrow \mathbb{R}$
$f(x) = (x-1)(x-2)\dots(x-11)$ | (p) one one |
| (B) $f: \mathbb{R} - \{-4/3\} \rightarrow \mathbb{R}$
$f(x) = \frac{2x+1}{3x+4}$ | (q) onto |
| (C) $f: \mathbb{R} \rightarrow \mathbb{R}$
$f(x) = e^{\sin x} + e^{-\sin x}$ | (r) many one |
| (D) $f: \mathbb{R} \rightarrow \mathbb{R}$
$f(x) = \log(x^2 + 2x + 3)$ | (s) into |

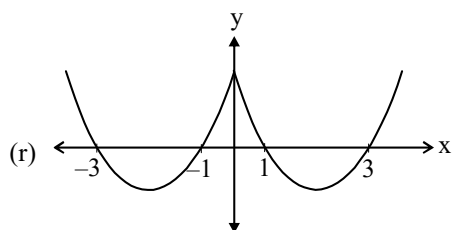
Following questions contains statement statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

2. Column-I

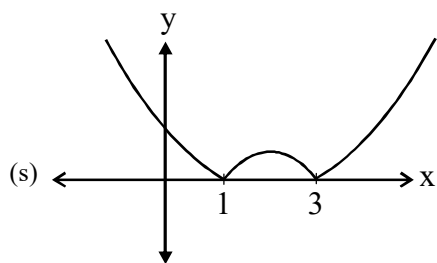
(A) If $f(x) = x^2 - 4x + 3$, then graph of $f(|x|)$ is

(B) If $g(x) = \frac{1}{\ln x}$, then its graph is

(C) If $f(x) = x^2 - 4x + 3$, then graph of $|f(x)|$ is



(D) If $k(x) = \frac{1}{\{x\}}$, then its graph is



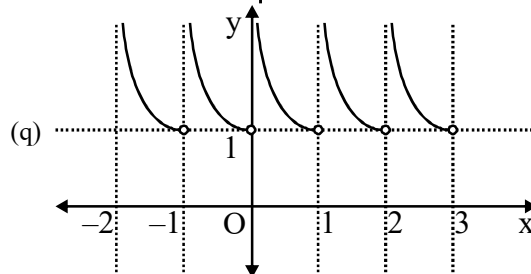
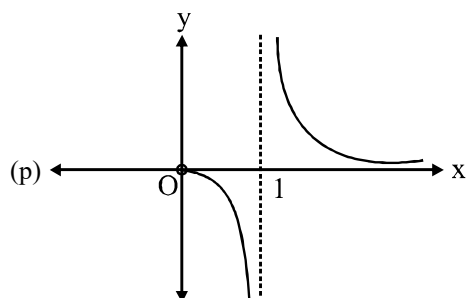
3. Column-I

(A) $\frac{\cos^2 x + \cos x + 2}{\cos^2 x + \cos x + 1}$

Column-II

(p) $\left(0, \frac{7}{3}\right)$

Column-II



(B) $\left| \frac{(\sqrt{\cos x} - \sqrt{\sin x})(\sqrt{\cos x} + \sqrt{\sin x})}{3(\cos x + \sin x)} \right|$ (r) $\left[\frac{4}{3}, \frac{7}{3}\right]$

(C) $\frac{7}{3(x^6 + 2x^4 + 3x^2 + 1)}$ (s) $\left[0, \frac{1}{3}\right]$

(D) $\log_8(x^2 + 2x + 2)$ (t) $[0, \infty)$

4. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be functions such that $f(g(x))$ is a one-one function.

Column I

Column II

- | | |
|------------------------------------|------------------------|
| (A) Then $g(x)$ | (p) must be one-one |
| (B) Then $f(x)$ | (q) may not be one-one |
| (C) If $g(x)$ is onto, then $f(x)$ | (r) may be many-one |
| (D) If $g(x)$ is into, then $f(x)$ | (s) must be many-one |

5. Column I: Function

Column II: Period

- | | |
|---|-----------------------------------|
| (A) $f(x) = \{(\text{sgn } x)^{\text{sgn } x}\}^n; x \neq 0,$
n is an odd integer | (p) odd function |
| (B) $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$ | (q) even function |
| (C) $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$ | (r) neither odd nor even function |
| (D) $f(x) = \max\{\tan x, \cot x\}$ | (s) periodic |

6. $\{.\}$ denotes the fractional part function and $[.]$ denotes the greatest integer function :

Column I : Function

Column II : Period

- (A) $f(x) = e^{\cos^4 \pi x + x - [x] + \cos^2 \pi x}$ (p) $1/3$
 (B) $f(x) = \cos 2\pi\{2x\} + \sin 2\pi\{2x\}$ (q) $1/4$
 (C) $f(x) = \sin 3\pi\{x\} + \tan \pi[x]$ (r) $1/2$
 (D) $f(x) = 3x - [3x + a] - b$, where $a, b \in \mathbb{R}^+$ (s) 1

8. **Column I**

Column II

- (A) $f : \mathbb{R} \rightarrow \left[\frac{3\pi}{4}, \pi \right)$ and (p) one-one

$$f(x) = \cot^{-1}(2x - x^2 - 2).$$

Then $f(x)$ is

- (B) $f : \mathbb{R} \rightarrow \mathbb{R}$ and (q) into

$$f(x) = e^{px} \sin q x \text{ where } p, q \in \mathbb{R}^+.$$

Then $f(x)$ is

- (C) $f : \mathbb{R}^+ \rightarrow [4, \infty)$ and (r) many-one

$$f(x) = 4 + 3x^2. \text{ Then } f(x) \text{ is}$$

- (D) $f : X \rightarrow X$ and $f(f(x)) = x \forall x \in X$, (s) onto

Then $f(x)$ is

9. **Column I : Function**

Column II : Domain

- (A) $f(\tan x)$ (p) $\left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2} \right], n \in \mathbb{Z}$

- (B) $f(\sin x)$ (q) $\left[2n\pi, 2n\pi + \frac{\pi}{6} \right] \cup$

$$\left[2n\pi + \frac{5\pi}{6}, (2n+1)\pi \right], n \in \mathbb{Z}$$

- (C) $f(\cos x)$ (r) $[2n\pi, (2n+1)\pi], n \in \mathbb{Z}$

- (D) $f(2\sin x)$ (s) $\left[n\pi, n\pi + \frac{\pi}{4} \right], n \in \mathbb{Z}$

10. **Column I : Function**

Column II : Period

- (A) $f(x) = \cos(|\sin x| - |\cos x|)$ (p) π

- (B) $f(x) = \cos(\tan x + \cot x)$ (q) $\pi/2$

$$\cos(\tan x - \cot x)$$

- (C) $f(x) = \sin^{-1}(\sin x) + e^{\tan x}$ (r) 4π

- (D) $f(x) = \sin^3 x \sin 3x$ (s) 2π

11. **Column I : Function**

Column II : Range

- (A) $f(x) = \log_3(5 + 4x - x^2)$ (p) Function not defined

- (B) $f(x) = \log_3(x^2 - 4x - 5)$ (q) $[0, \infty)$

- (C) $f(x) = \log_3(x^2 - 4x + 5)$ (r) $(-\infty, 2]$

- (D) $f(x) = \log_3(4x - 5 - x^2)$ (s) \mathbb{R}

TYPE 2

Given below are Matching Type Questions, with two columns (each having some items) each. Each item of Column I has to be matched with the items of Column II, by encircling the correct matches.

1. **Column I**

Column II

- (A) Let $f(x) = \max\{1 + \sin x, 1, 1 - \cos x\}$, (P) $g(f(1)) = 1$

$$x \in [0, 2\pi] \text{ and } g(x) = \max\{1, |x - 1|\},$$

$x \in \mathbb{R}$ then

- (B) Let $f(x) = \ln\left(\frac{1+x}{1-x}\right) \forall x \in (-1, 1)$ (Q) $f(g(0)) = 0$

$$\text{and } g(x) = \left(\frac{3x + x^3}{1 + 3x^2}\right), \text{ then}$$

- (C) Let $f(x) = 1 + x^2$ and $g(x) = x - x^2$, (R) $f(g(0)) = 1$

then

(S) $g(f(0)) = 1$

(T) $g\left(f\left(\frac{e-1}{e+1}\right)\right) = 1$

2. **Column I**

Column II

- (A) If $y = 2|x - 2| + 3|x + 1|$, then (P) $y = \begin{cases} -5x + 1, \\ x + 7, \end{cases}$

$$\begin{matrix} x < -1 \\ -1 \leq x < 2 \end{matrix}$$

- (B) If $y = 2|x - 2| - 2|x + 1|$, then (Q) $y = \begin{cases} 5x - 1, \\ x + 7, \end{cases}$

$$\begin{matrix} -1 \leq x < 2 \\ x \geq 2 \end{matrix}$$

- (C) If $y = 3|x + 1| - 2|x - 2|$, then (R) $y = \begin{cases} x + 7, \\ -5x + 1, \end{cases}$

$$\begin{matrix} x < -1 \\ -1 \leq x < 2 \end{matrix}$$

(S) $y = \begin{cases} x + 7, \\ 5x - 1, \end{cases}$

$$\begin{matrix} -1 \leq x < 2 \\ x \geq 2 \end{matrix}$$

(T) $y = \begin{cases} -x - 7, \\ 5x - 1, \end{cases}$

$$\begin{matrix} x < -1 \\ -1 \leq x < 2 \end{matrix}$$

3. **Column I**

Column II

- (A) The inverse of $\sin(\tan^{-1} x)$ is (P) $\cot(\cos^{-1} x)$

- (B) The inverse function of

(Q) $\frac{\log_2 x}{\log_2 x - 1}$

$$f(x) = 1 - 2^{-x} \text{ is}$$

- (C) The inverse function

(R) $\tan(\sin^{-1} x)$

$$f(x) = 2^{x/(x-1)}$$

(S) $-\log_2(1-x)$

(T) $\frac{x}{\sqrt{(1-x^2)}}$

$$f(x) = \frac{1}{\sqrt{\{\sin x\} + \{-\sin x\}}} \text{ is } \dots$$

(where $\{.\}$ denotes the fractional part function)

4. Column I

(A) The range of the function

$$f(x) = \frac{1}{2 - \cos 3x} \text{ is}$$

(B) The range of the function

$$f(x) = \sqrt{(x-4)} + \sqrt{(6-x)} \text{ is (Q) a subset of } [1, 2]$$

(C) If the function $f(x)$ is

(R) $\left[-\frac{3}{2}, -1\right]$

defined for $x \in [0, 1]$, then the function $f(2x+3)$ is defined for

(S) $\left[\frac{1}{3}, 1\right]$

(T) $[\sqrt{2}, 2]$

5. Column I

(A) The function $f(x) = (x - [x])^2$, (where $[x]$ is greatest integer function $\leq x$) is

(B) The function

$$f(x) = \log_a(x + \sqrt{(x^2 + 1)}); a > 0, a \neq 1, \text{ is (Q) non-periodic}$$

(Assume it to be an onto)

(C) The function $f(x) = \cos(5x+2)$ is

Column II

(P) periodic

(R) one-one

(S) many one

(T) invertible

[PART 4 - FILL IN THE BLANKS]

1. The number of bijective function from set A to itself when A contains 106 elements is

2. The period of the function $f(x) = \cos(\tan x + \cot x)$. $\cos(\tan x - \cot x)$ is

3. $e^{f(x)} = \frac{10+x}{10-x}$, $x \in (-10, 10)$ and $f(x) = \lambda f\left(\frac{200x}{100+x^2}\right)$, then $\lambda = \dots$

4. The function $f(x)$ is defined for all real x . If $f(a+b) = f(ab)$ $\forall a$ and b and $f\left(-\frac{1}{2}\right) = -\frac{1}{2}$, then $f(2012) = \dots$

5. The domain of definition of the function

6. If $f(x) = \ln\left(\frac{x^2+e}{x^2+1}\right)$, then range of $f(x)$ is

7. If f is an even function defined on the interval $(-5, 5)$, then four real values of x satisfying the equation

$$f(x) = f\left(\frac{x+1}{x+2}\right) \text{ are } \dots, \dots, \dots \text{ and } \dots$$

8. If $F(n+1) = \frac{2F(n)+1}{2}$, $n = 1, 2, 3, \dots$ and $F(1) = 2$, then $F(101) = \dots$

9. Suppose that $g(x) = 1 + \sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x$, then $f(x)$ is

10. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ such that $\min f(x) > \max g(x)$, then the relation between b and c is

11. If $\sum_{k=0}^n f(x+ka) = 0$, where $a > 0$, then the period of $f(x)$ is

12. There are exactly two distinct linear functions, and which map $[-1, 1]$ on to $[0, 2]$.

13. The domain of $f(x) = \sqrt{1 - \sqrt{1 - \sqrt{(1-x^2)}}}$ is

14. If $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$, then $(g \circ f)(x)$ is invertible in the domain.

15. If $b^2 - 4ac = 0$, $a > 0$, then the domain of the function $f(x) = \log_c[ax^3 + (a+b)x^2 + (b+c)x + c]$ is

16. If $g : [-2, 2] \rightarrow \mathbb{R}$ where $g(x) = x^3 + \tan x + \left[\frac{x^2+1}{\lambda}\right]$ is an odd function, then the value of parameter λ is

17. The period of the function $f(x) = \frac{1}{3}(\sin 3x + |\sin 3x| + [\sin 3x])$ where $[.]$ denotes the greatest integer function, is.

18. Range of $f(x) = 3 \tan \sqrt{\left(\frac{\pi^2}{9} - x^2\right)}$ is

19. Let $f(x) = \begin{cases} 1 + [x], & x < -2 \\ |x|, & x \geq -2 \end{cases}$ (where $[.]$ denotes the greatest integer function), then $f(f(-2.6))$ is

20. Let $f(x) = \begin{cases} 4, & x < -1 \\ -4x, & -1 \leq x < 0 \end{cases}$, then its even extension is.

[PART 4 - INTEGER TYPE QUESTION]

1. If a, b and c are non-zero rational numbers, then the sum of all the possible values of $\frac{|a|}{a} + \frac{|b|}{b} + \frac{|c|}{c}$ is _____

2. The number of integral values of a for which $f(x) = \log(\log_{1/3}(\log_7(\sin x + a)))$ is defined for every real value of x is

3. Let f be a real valued invertible function such that $f\left(\frac{2x-3}{x-2}\right) = 5x-2, x \neq 2$. Then the value of $f^{-1}(13)$ is

4. The number of integral values of x for which the function $\sqrt{\sin x + \cos x} + \sqrt{7x - x^2 - 6}$ is defined is

5. If θ is the fundamental period of the function $f(x) = \sin^{99}x + \sin^{99}\left(x + \frac{2\pi}{3}\right) + \sin^{99}\left(x + \frac{4\pi}{3}\right)$, then the complex number $z = |z|(\cos\theta + i \sin \theta)$ lies in the quadrant number. _____

6. The number of values of x for which $||x^2 - x + 4| - 2| - 3| = x^2 + x - 12$ is

7. The number of integers in the domain of function, satisfying $f(x) + f(x^{-1}) = \frac{x^3 + 1}{x}$, is

8. If $4^x - 2^{x+2} + 5 + ||b - 1| - 3| = |\sin y|$, $x, y, b \in \mathbb{R}$, then the possible value of b is _____

9. A function f from integers to integers is defined as $f(x) = \begin{cases} n+3, & n \in \text{odd} \\ n/2, & n \in \text{even} \end{cases}$ suppose $k \in \text{odd}$ and $f(f(f(k))) = 27$. Then the sum of digits of k is

10. Let $f(x) = 3x^2 - 7x + c$, where c is a variable coefficient and $x > \frac{7}{6}$. Then the value of [c] such that $f(x)$ touches $f^{-1}(x)$ is (where [.] represents greatest integer function)

11. Suppose that $f(x)$ is a function of the form $f(x) = \frac{ax^8 + bx^6 + cx^4 + dx^2 + 15x + 1}{x}, (x \neq 0)$. If $f(5) = 2$, then the value of $|f(-5)|/4$ is

12. If $x = \frac{4}{9}$ satisfies the equation $\log_a(x^2 - x + 2) > \log_a(-x^2 + 2x + 3)$, then the sum of all possible distinct values of [x] is (where [.] represents the greatest integer function).

13. The number of integral values of x satisfying the inequality $\left(\frac{3}{4}\right)^{6x+10-x^2} < \frac{27}{64}$ is

14. If $f(x) = \sqrt{4-x^2} + \sqrt{x^2-1}$, then the maximum value of $(f(x))^2$ is

15. The function f is continuous and has the property $f(f(x)) = 1 - x$. Then the value of $f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)$ is

16. The number of integers in the range of the function $f(x) = \left| 4 \frac{(\sqrt{\cos x} - \sqrt{\sin x})(\sqrt{\cos x} + \sqrt{\sin x})}{(\cos x + \sin x)} \right|$ is

17. If T is the period of the function $f(x) = [8x + 7] + |\tan 2\pi x + \cot 2\pi x| - 8x$ (where [.] denotes the greatest integer function), then the value of 1/T is

18. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$. If N is the number of onto functions from E to F, then the value of $N/2$ is

19. If $f(x)$ is an odd function, $f(1) = 3$ and $f(x+2) = f(x) + f(2)$, then the value of $f(3)$ is

20. Let $a > 2$ be a constant. If there are just 18 positive integers satisfying the inequality $(x-a)(x-2a)(x-a^2) < 0$, then the value of a is.

ANSWER KEY DPP

DPP 1

1. $\left\{x : x = \frac{n}{n+1}, \text{ where } n \text{ is a natural number and } 1 \leq n \leq 6\right\}$
2. (i) \rightarrow (D) (ii) \rightarrow (C) (iii) \rightarrow (A) (iv) \rightarrow (B) 3. (C) 4. (B) 5. (A)
6. (i) We have, $X = \{A, L, L, O, Y\}$, $B = \{L, O, Y, A, L\}$. Then X and B are equal sets as repetition of elements in a set do not change a set. Thus, $X = \{A, L, O, Y\} = B$
(ii) $A = \{-2, -1, 0, 1, 2\}$, $B = \{1, 2\}$. Since $0 \in A$ and $0 \notin B$, A and B are not equal sets.
7. No. $A \not\subset B$ as $i \in A$ and $i \notin B$
No. $B \not\subset A$ as $d \in B$ and $d \notin A$
8. (D) 9. (A) 11. $A - B = \{1, 3, 5\}$, $B - A = \{8\}$

DPP 2

1. $(-2, \infty)$ 2. $x \in [8, \infty)$ 3. $x < 3$ 4. $x \in (2, 3)$ 5. $x \in (-\infty, 1) \cup (2, 3)$
6. $-1 \leq x < 2$ 7. $\frac{-11}{3} \leq x \leq 5$ 8. $x \in (-\infty, -1) \cup (0, 1/2) \cup (1 - \infty)$ 9. $x \in (-\infty, -3) \cup (-2, -1)$
10. $x \in (-\infty, -9) \cup (-9, -3) \cup [-1, 0) \cup (0, 2) \cup [4, 6)$

DPP 3

1. (B) 2. (C) 3. (C) 4. (D) 5. (C) 6. (A)
7. (C) 8. (C) 9. (B) 10. (B)

DPP 4

1. $\left[-\frac{13}{4}, \infty\right)$ 2. $[\sqrt{2}, \infty)$ 3. $(-\infty, 5] \cup [9, \infty)$ 4. $[0, 2]$ 5. (D)
6. $f(A) = \{0, -2, 18, 28, 108\}$, $f(A) \neq B$ 7. $D_f = \{\phi\}$ 8. (D) 9. (C, D)
10. (B, D)

DPP 5

1. $2n\pi - \frac{\pi}{6} < x < 2n\pi + \frac{7\pi}{6}$; $n \in \mathbb{Z}$ 2. $\left(-\infty, -\frac{1}{3}\right] \cup [1, \infty)$ 3. $1 \leq f(x) \leq \sqrt{2}$
4. $x \in \left(-\frac{1}{6}, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 6\right)$ 5. (i) $\cos [2, 1]$ (ii) $[-1, 1]$ (iii) $[-1, 1]$
- (iv) $[0, \infty)$ (v) $[-\sqrt{2}, \sqrt{2}]$

DPP 6

1. $x \in (-\infty, -1) \cup [0, \infty)$ 2. $(8, 10)$ 3. $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ 4. $x \in (-\infty, -6] \cup [-2, 4] \cup [8, \infty)$
5. $x \in (-5, -2) \cup (-1, \infty)$ 6. $x \in \left[0, \frac{1}{\sqrt{2}}\right]$ 8. $x \in [1, 4]$ 9. $x \in (1, \infty) \cup \{-1\}$
10. (i) $\log\left(\frac{4}{5}\right) \leq \log(5x^2 - 8x + 4) < \infty$ (ii) $(-\infty, 2]$

DPP 7

2. Domain is $\mathbb{R} - I$, Range is $(-\infty, 0)$. 3. $\{0\}$ 4. $0 \leq 1 - \frac{1}{1+\{x\}} < \frac{1}{2}$
5. (i) $x \in \mathbb{R} - \left\{-\frac{1}{2}, 0\right\}$ (ii) $x \in (-2, -1) \cup (-1, 0) \cup (1, 2)$ 6. $x \in (-\infty, 1] \cup [12, \infty) - \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
7. Range = $\{1, -1, 0\}$

DPP 8

4. onto
6. (a) $f(x)$ is many-one and neither surjective nor injective.
(b) Function is onto Many one and Surjective but not injective
(c) Function is many one and neither injective nor surjective.
7. $p \rightarrow (b, d), q \rightarrow (a, c), r \rightarrow (a, c), s \rightarrow (b, d), t \rightarrow (a, c)$
8. $f(2) = f(3^{1/4}) \Rightarrow$ many-to-one function
 $f(x) f(x) \neq \sqrt{3} \forall x \in \mathbb{R} \Rightarrow$ into function
9. 181 10. 1561

DPP 9

1. $-\frac{1}{2}$ 2. 217 3. $f(x) = \frac{4004}{x} - x$ 4. $f(x) = 1 - \frac{x^2}{x}$
6. 997.5 7. (A) 8. -1 9. 64 10. 5050

DPP 10

1. $c = 1$ and d is arbitrary. 2. (i) Odd (ii) Neither even nor odd (iii) Even
3. Neither odd nor even 4. $x = \frac{-3 \pm \sqrt{5}}{2}$ 5. Odd 6. Odd, $f(0) \neq 0$
7. (i) Even (ii) Even (iii) Even (iv) Odd 8. 2014

DPP 11

1. (i) Invertible (ii) Not invertible
2. $g(x) = \frac{x+8}{3}$ 3. $\frac{x-5}{3}$ 4. $f^{-1}(x) = \frac{1}{2}(a^x - a^{-x})$ 5. $\frac{\pi}{2}$
6. 1 8. (i) 4π (ii) Not periodic (iii) $\frac{\pi}{2}$ (iv) 2π
9. 4π 10. 15 11. 1 12. Periodic Function
13. 2 14. Periodic

EXERCISE - I

1. (C) 2. (A) 3. (D) 4. (B) 5. (C) 6. (B) 7. (B) 8. (C) 9. (A) 10. (D)
11. (C) 12. (B) 13. (B) 14. (B) 15. (C) 16. (B) 17. (D) 18. (B) 19. (B) 20. (A)
21. (C) 22. (B) 23. (B) 24. (A) 25. (C) 26. (C) 27. (B) 28. (A) 29. (C) 30. (B)
31. (C) 32. (C) 33. (D) 34. (D) 35. (B) 36. (C) 37. (C) 38. (D) 39. (A) 40. (D)
41. (B) 42. (C) 43. (B) 44. (A) 45. (D) 46. (A) 47. (D) 48. (C) 49. (C) 50. (B)
51. (B) 52. (D) 53. (B) 54. (D) 55. (C) 56. (B) 57. (D) 58. (C) 59. (B) 60. (B)
61. (D)

EXERCISE - II

1. (B, C)
2. (A, B, C)
3. (B, C)
4. (A, B, C)
5. (A, B, C, D)
6. (A, B, C, D)
7. (B, D)
8. (A, B, C)
9. (A, C)
10. (B, D)
11. (B, D)
12. (A, D)
13. (A, C)
14. (B, C)
15. (A, B, D)
16. (A, B, C, D)
17. (B, C)
18. (A, B)
19. (C)
20. (A)
21. (A, C, D)
22. (B, D)
23. (B)
24. (A, D)
25. (B, C)
26. (A, B, C)
27. (A, B)
28. (B, C, D)
29. (A, B, C, D)
30. (A, D)
31. (A, B)
32. (A, B, C)
33. (A, D)
34. (B, C)
35. (A, B, C)
36. (B, C, D)
37. (A, B)
38. (A, B, C)
39. (A, B, C)
40. (A, B, D)
41. (A, B, C, D)
42. (C, D)

EXERCISE - III

PART - I

Comprehension 1

1. (C)
2. (D)
3. (D)
4. (D)
5. (D)

Comprehension 3

11. (C)
12. (C)
13. (B)
14. (C)
15. (A)

Comprehension 5

21. (C)
22. (B)
23. (C)
24. (C)
25. (C)

Comprehension 7

29. (C)
30. (C)
31. (B)

Comprehension 9

35. (D)
36. (D)
37. (C)

Comprehension 11

41. (B)
42. (C)
43. (D)

Comprehension 13

47. (A)
48. (D)
49. (C)

Comprehension 15

53. (C)
54. (C)
55. (C)

Comprehension 2

6. (C)
7. (B)
8. (A)
9. (B)
10. (C)

Comprehension 4

16. (B)
17. (A)
18. (A)
19. (A)
20. (B)

Comprehension 6

26. (C)
27. (C)
28. (B)

Comprehension 8

32. (A)
33. (B)
34. (A)

Comprehension 10

38. (D)
39. (C)
40. (C)

Comprehension 12

44. (A)
45. (B)
46. (C)

Comprehension 14

50. (D)
51. (C)
52. (C)

PART - II

Assertion Reason

1. (A)
2. (C)
3. (D)
4. (B)
5. (D)
6. (B)
7. (A)
8. (A)
9. (C)
10. (D)
11. (A)
12. (A)
13. (B)
14. (B)
15. (B)
16. (B)
17. (A)
18. (A)
19. (D)
20. (D)
21. (A)
22. (D)
23. (A)
24. (A)
25. (A)
26. (D)
27. (A)
28. (B)
29. (D)

PART - III

Match the Column Type

1. (A) \rightarrow (r, q); (B) \rightarrow (p, s); (C) \rightarrow (r, s); D \rightarrow (r, s)
2. (A) \rightarrow (r); (B) \rightarrow (p); (C) \rightarrow (s); (D) \rightarrow (q)
3. (A) \rightarrow (q); (B) \rightarrow (r); (C) \rightarrow (p); (D) \rightarrow (s)
4. (A) \rightarrow (p); (B) \rightarrow (q, r); (C) \rightarrow (p); (D) \rightarrow (q, r)
5. (A) \rightarrow (p); (B) \rightarrow (q); (C) \rightarrow (q, s); (D) \rightarrow (p, r)
6. (A) \rightarrow (s); (B) \rightarrow (r); (C) \rightarrow (s); (D) \rightarrow (p)
7. (A) \rightarrow (r, s); (B) \rightarrow (r, s); (C) \rightarrow (p, q); (D) \rightarrow (p, s)
8. (A) \rightarrow (s); (B) \rightarrow (r); (C) \rightarrow (p); (D) \rightarrow (q)
9. (A) \rightarrow (q); (B) \rightarrow (q); (C) \rightarrow (s); (D) \rightarrow (p)
10. (A) \rightarrow (r); (B) \rightarrow (s); (C) \rightarrow (q); (D) \rightarrow (p)

PART - IV

Fill in the Blanks

1. 106! 2. $\pi/2$ 3. 1/2 4. -1/2 5. $R - \frac{n\pi}{2}, n \in I$ 6. (0, -1]
7. $x = \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$ 8. 52 9. $2+x^2$ 10. $|c| > \sqrt{2}|b|$
11. $(n+1)a$ 12. $x+1$ or $-x+1$ 13. R 14. $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
15. $R \sim \left\{(-\infty, -1) \left\{-\frac{b}{2a}\right\}\right\}$ 16. $\lambda \in (5, \infty)$ 17. $\frac{2\pi}{3}$
18. $[0, 3\sqrt{3}]$ 19. 2 20. $f(x) = \begin{cases} 4x, & 0 < x \leq 1 \\ 4, & x > 1 \end{cases}$

PART - V

Integer Type

1. (0) 2. (3) 3. (3) 4. (3) 5. (3) 6. (1) 7. (2) 8. (4) 9. (6)
10. (5) 11. (7) 12. (1) 13. (7) 14. (6) 15. (1) 16. (5) 17. (4) 18. (7)
19. (9) 20. (5)

EXERCISE - IV

1. (A) 2. (B) 3. (A) 4. (B) 5. (B) 6. (D) 7. (D) 8. (C) 9. (B)
10. (B) 11. (C) 12. (D) 13. (A) 14. (D) 15. (A) 16. (D) 17. (A) 18. (A)
19. (D) 20. (C) 21. (B) 22. (C) 23. (B) 24. (A) 25. (C) 26. (B) 27. (B)
28. (B) 29. (B) 30. (D) 31. (3) 32. (i) (B) (ii) (A) (iii) (D) (iv) (A) (v) (D)
33. (i) (D) (ii) (A) 34. (i) (C) (ii) (B) 35. (B) 36. (A) 37. (A) 38. (A)
39. (B) 40. (B) 41. (A, B, C) 42. (D) 43. (D) 44. (D) 45. (D)
46. (C) 47. (A)

MOCK TEST

1. (A) 2. $x \in [1, 2) \cup (2, 3]$ 3. $1 \leq \log_2 \left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}} \right) \leq 2$ 4. $x \in [3/2, 2]$
5. $x \in [2, \infty)$ 6. 107 7. [-1, 3] 8. (B)
9. $f(n) = (n+1)^2$ 10. $f(x) = 1 - \frac{x^2}{2}$ 11. $2x-3$ or $-2x+2$ 12. [0, 1]
13. $n \in \left(n\pi - \frac{\pi}{2}, n\pi + \tan^{-1} 2 \right)$ 14. $f^{-1}(x) = x+2$
15. (a) periodic function (b) periodic function (c) periodic function (d) non-periodic
16. $n \in [9, 10)$ 17. There are exactly six solutions. 18. (B) 19. (A)
20. (C) 21. (A, B) 22. (B)
23. (A, B, C, D) 24. (A) 25. $a \rightarrow p; b \rightarrow q; c \rightarrow q, s; d \rightarrow p, r$
26. (D) 27. (C) 28. (C) 29. (0) 30. (4)